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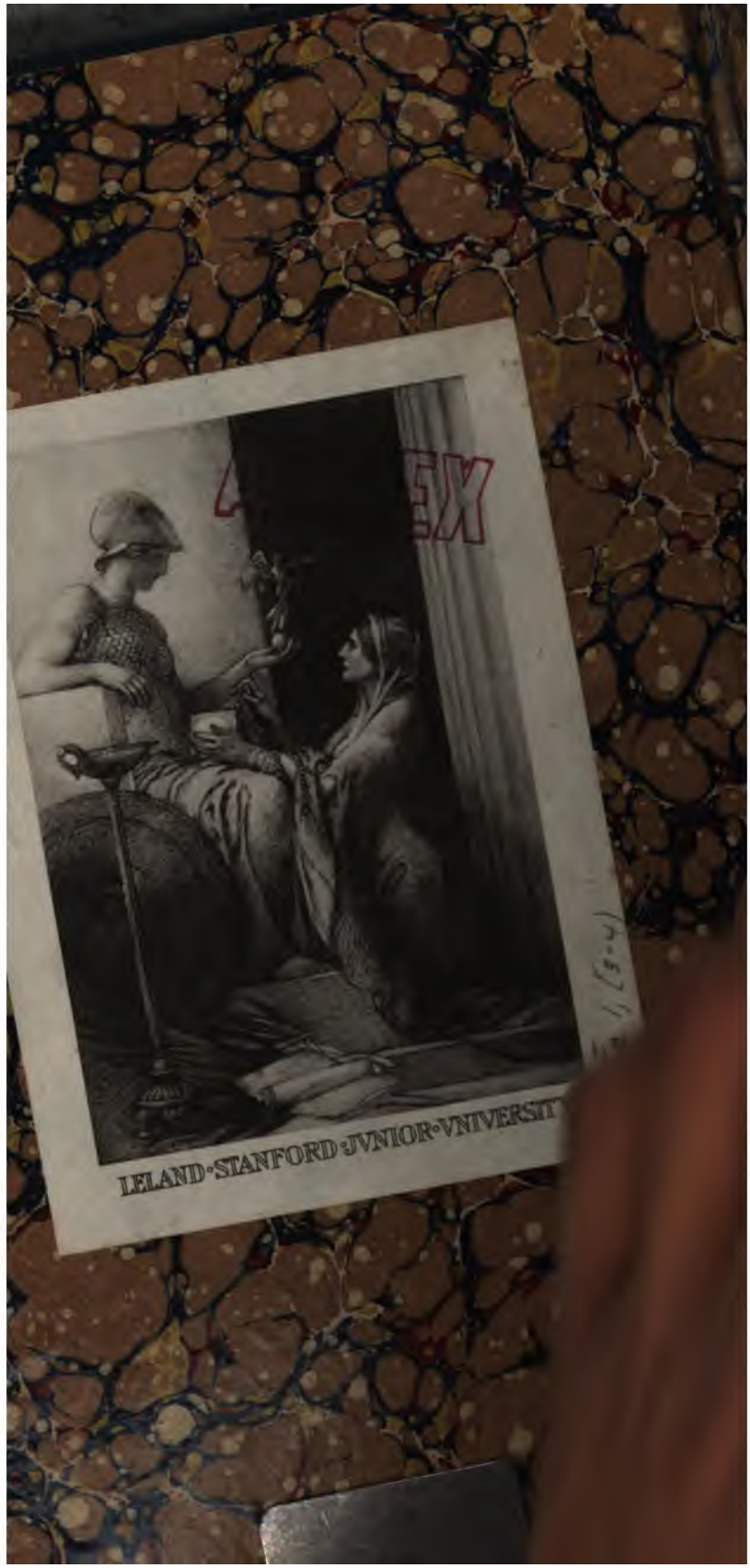
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VOLUME XXVI

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and while common men and common women on the fields of battle, at sea and in the houses of sorrow, are showing themselves much better than our schools, our parliaments, our churches could ever have hoped to make them, the proud of the earth, the intellectual, the spiritual, stand before us, revealed in their shameful and piteous nakedness. We know from their utterances in these times of stress that the whole of their sham activity has always been nothing but an impotent effort to express in rational terms the feelings and the desires that make life possible and lovable to them, and that the field of their vision is not any wider or clearer than that of the unborn child in the warm and moist darkness of the maternal womb.

I have said "the value of human thought": I should have said the value of official, recognized, representative thought and religion. There have been, and perhaps there are even now, some men for whom the business of thought has not been a pretence and a fraud, but as much a true life and an austere militancy as that of the soldier and sailor; some men for whom thought has been their true life and not a quarry of commonplaces for the justification of a different and lower life. Besides the philosophy of the schools and the religion of the churches, there are in the history of mankind a few isolated individuals who have faced life only with the strength of their unaided soul; it is from them that the schools and the churches have stolen the sacred spark to kindle their comfortable little fires; it is to them that we may still look for an example when everything else has failed us. Philosophy is then no more to us the prearranged cycle of reality unfolding itself in thought, of the Hegelian, or the unborn science that the realist affectionately nurses; but just a collection of a few supreme views of life, of a few equations of individual souls to infinite truth, of a few definite truths strangely resembling each other though varying in form and often

apparently contradictory. Philosophy is then nothing if not ethical and personal.

II.

The work of Carlo Michelstaedter answers these two requirements. It is personal ethics. If I had to find a place for it in one of those ludicrous puppet-shows called histories of philosophy, I am afraid it would refuse to be put anywhere or would turn out to be tremendously anachronistic. It refuses to be explained by any shuffling of *Kulturgeschichte*.

Although his name was German and his blood, I think, partly Jewish, he came to us from Gorizia, a Venetian town of unredeemed Italy, as much an Italian as any man can be. And he wrote a beautiful, nervous and supple prose of his own that, if into any language, could be properly translated only into Greek. His thoughts seemed to find their natural medium of expression in Greek, and Greek phrases and words and particles are continually interwoven in his Italian prose like an efflorescence of a deeper self. He had certainly read much, but he had a wonderful power of going back from the complicated superstructures of intellect to the elemental realities of life: and in the purity and directness of his speculation he had very few helpers and companions: Plato before his senile decadence and in his faithfulness to Socrates; Parmenides and Heraclitus among the pre-Socratics; Æschylus, Sophocles, and our Greek Leopardi among the poets; Christ, who talks Greek in the New Testament. From the dark depths of the sacred books one kindred voice responded to him in the solemn music of Ecclesiastes.

No more need be said about his culture but that before his mind was completely absorbed by his work he had given some years of his life to mathematics and had emerged from them, as I think, with empty hands but with a sharper

sight. His life is summed up in a few words. He was a student of *literae humaniores* at the University of Florence; he committed suicide in his native town of Gorizia in his twenty-third year, at the end of October 1910. Before deliberately sending a bullet through his heart he had accomplished his *opus* and sent it in a spirit of bitter scorn as his thesis for the degree to his university. He was a good climber of mountains, a keen sailor, a healthy and handsome young man. He had in Florence a small band of adoring friends, and wealth and the love of women. He did not commit suicide through wretchedness and despair or in a cowardly fear of what life might have brought to or withdrawn from him; his action seemed to him to be the necessary development of his thought, the highest affirmation of his life. His suicide was a purely metaphysical, or ethical, one. I am sorry for those who delight in explaining away what is unpalatable to them in thought by finding particular reasons and motives in the psychology of the thinker, but although I cannot say that I entirely feel that his truth is my truth (and which truth is *the* truth?), I understand when I read him that the man who has said what he said cannot go on living—in the same sense in which the Lord said to Moses: "Thou canst not see my face; because man cannot see me, and live."

The writings of Michelstaedter have been edited by his friend Vladimiro Arangio-Ruiz in two slender volumes (*Scritti di Carlo Michelstaedter,* Genova, 1912-13) of not more than two hundred and fifty pages altogether. The first volume comprises his *Dialogo della Salute* and a few lyrical poems; the second his thesis, *La Persuasione e la Rettorica,* where the ideas rapidly and dramatically expressed in the dialogue are more fully developed and expounded. A third volume which has not yet appeared will contain appendices and explanatory notes to the matters treated in the second.

It is not an easy task to give in brief the main lines of Michelstaedter's thought. He wrote so little that nothing is to be found in his pages that might be considered as superfluous. They stand before us in the white heat of an all-pervading moral enthusiasm that gives to each of them an unforgettable physiognomy and a meaning necessary to the understanding of every other. But although quoting from them is to me like tearing a beautiful organism asunder I must try to define his position and I shall do so as far as possible by means of extracts and in his own words. What I am interested in now is in fact only to make his words known to those who have not yet heard as much as his name.

III.

I know that I will, and there is nothing for me to will. A weight hangs from a hook, and as it hangs it suffers in that it cannot descend: it cannot come off the hook, as it hangs owing to the fact that it is a weight, and its weight is its dependence. We want to satisfy it: we free it from its dependence, we let it go, that it may satisfy its hunger for a lower point and descend independent until it is satisfied with descending. But at each of the points it successively reaches it will never be contented to stop and it still wishes to descend, for the next point is always lower than the one it occupies. And none of the future points will ever be such as to satisfy it, each of them being necessary to its life only as long as a lower point awaits it, ὅφρα ἄν μένῃ αὐτόν; but every time, each point, when it is made actual, will become for it devoid of every attraction, no longer being a lower point; so that *in each point it feels the want of the lower points* and more and more do these attract it. It always has the same hunger for the lower, and its will to descend remains infinite, since, if the will could become finite in any particular point and the weight

could there obtain the infinite descent in the infinite future, in that point it would no longer be what it is, a weight.

Its life consists in this lacking in life. If it did not want anything but was *finite* and could possess itself in its perfection it would have ceased to exist. The weight is to itself a hindrance to possessing its life, and it is only because of itself that it is never able to be satisfied. The weight can never be *persuaded* (II, 1-2).

And no life is ever sated with living in any present, for it is life only inasmuch as it continues itself, and it continues itself in the future only inasmuch as it lacks living. If it possessed itself fully, now and here, and lacked nothing, if nothing awaited it in the future, it would not continue itself: it would cease to be life (II, 2).

What we call life is this perpetual deficiency through which everything that lives dies at every instant continuing itself and which at every point expresses itself in the will of determinate things entering into some relation with some other thing. Every thing at every point does not possess any other thing but is a will of a determinate possession, that is, a determinate attribution of value, a determinate consciousness. While entering into a relation with the given thing it believes to be in the act of possession, and is nothing but a determinate power, a *finita potestas*. Every present, every actuality, what at any time, under any condition, we call life, is the infinitely variable conjunction of the powers determinately localized in the infinitely variable aspects, as a consciousness to which its correlate is every time stable in its own instability. There is nothing that is *per se*, but only in relation to a consciousness. Life is therefore an infinite correlativity of consciousness (II, 6, 7).

An organic life is a complex of wills of determinate things. The organism determines itself successively in relation to the several things; but to every single determina-

tion is inherent the sense that it takes place not *per se* but only because it is necessary to the continuation of the organism. Herein is the sweet taste that every thing has in life and such is the voice of all other things toward which the organism shall determine itself in the future. Inasmuch as a thing is pleasant the whole self is in it *in actu*. And as it strives for the thing as for its possession it extracts from the thing the illusion of individuality. What I like, what is useful to me,—this is my consciousness, this is my reality. So reality ὀνομάζεται καθ' ἡδονὴν ἑκάστου, is named according to each man's pleasure. Every individual that wills to be, and instead of that becomes in time, is *actual* in every instant with the whole of its will. Pleasure and pain are the sum of life. Every act tells life: *"Thou art"* (I, 17, 18).

Every act helps man to build up his fictitious self; so that each time, in the actuality of his affirmation, he feels himself above and distinct from the present instant, and from the relation that belongs to that instant. He feels himself always the same in different times and in relation to different things; he says, *"I am"* (II, 14).

Such is the process of constitution of our illusory individuality, such is the inadequate persuasion that rules our life. Everybody knows as much as he wills, sees as much as he lives, as much of the distant as his pleasure makes near to him. But then he calls his world, which is nothing but his correlate, *the* world, and his volition of himself in the future, the end, the *raison d'être*, the meaning of his single acts (II, 16, 17).

His power over the things is at every point limited by the limited prevision of the satisfaction of his particular need. From the relation with the thing, he does not get the possession of the thing, but merely the security of his own life; but even this is soon at an end and the narrowness of his horizon is actual in every point through the

his knowledge—whatever it may be. Through his illusion he says that what is for him, *is*; he says it is good or bad, according to his liking or disliking it. But when he says, 'I know that this is,' he affirms himself in contraposition to an element of reality which is nothing but the affirmation of his own self; he wills himself willing, he puts his self in one of its affirmations as being real outside himself (II, 63-64).

Now for himself man either knows or does not know, but he says that he knows for others. His knowledge is in life, for life, but when he says "I know," he says to the others that he is alive, in order to get from the others something that is not given to him through his affirmation of life. He wants to constitute for himself an absolute self. That is what Michelstaedter calls "Rhetoric" and defines as the inadequate affirmation of individuality. The rhetoric of pleasure is but one of its particular forms, perhaps the first and most elementary. The rhetoric of knowledge follows it: the philosopher and the scientist are in a sense younger than the *gourmet* and the *viveur*.

As a child in the darkness shouts to give himself a sign of his own self which he feels to be failing in the infinite fear; so men who feel to be failing in the solitude of their empty souls, affirm themselves inadequately by simulating the sign of the self they do not possess, the knowledge, as if it were already in their hands. They hear no longer the voice of things that says to them: *'thou art,'* and in the darkness every one seeks the hand of his companion and says: 'I am, thou art, we are,' in order that the other may mirror him and say to him: 'thou art, I am, we are'—and they repeat together: 'we are, because we know, because we can say to each other the words of knowledge, of free and absolute knowledge.' As they have nothing and can give nothing, they take refuge in words that feign a communication. As they cannot each of them

make of his world the world of the other they simulate words containing the absolute world, and with words they feed their *ennui,* words they apply as a palliative to their pain; in words they express what they do not know and what they need in order to mitigate their pain. Every word contains the mystery, and in words they trust, with words they weave a new veil for the darkness, καλλωπίσματα ὄρφνης. "God help me"—as I have not the courage to help myself (II, 66).

They want knowledge, and knowledge is constituted. Knowledge is by itself the aim of life. There are the parts of knowledge, and the road to knowledge, and men who give it. It is bought and sold at a given price, in a given time, with a given amount of work. So rhetoric flourishes by the side of life: man puts himself in the position of knowledge, *posizione conoscitiva,* and makes knowledge. But as knowledge is in that way a necessity it is necessary also that there should always be a demand for it. If it were otherwise, the men who know, for whom should they know? What would a nurse be if there were no patients? And what a strange creature would then the physician be! But the patients are created. When the young spread their wings to rise from the accustomed life, when the cry of life, strange and obscure to themselves, bursts from their heart, when they ask to be truly men,—that is nothing, they say, but *thirst for knowledge.* And with the water of knowledge they quench their flame. The end, the *raison d'être,* and freedom and justice and possession, —everything is given to them in finite *words* which are applied to different things and then abstracted from them. If in everything they ask for life, of everything is given to them, in answer to their demand, the ὄνομα ἐπίσημον, the name that stands for a conventional sign. In this sign, through this convention, they presume they have the knowledge, each time a small piece of knowledge which, joined

with and subordinated to other pieces through the wonderful concatenation of philosophical curiosity, may form a system of names and constitute for them the inviolable possession of *absolute knowledge* (II, 67, 68).

The position of the knower is analogous to that of the *viveur*; the *viveur* craves for the sweetness of pleasure independently from the necessary relation which is the only source of pleasure, the knower feigns to himself an absolute life in the elaboration of knowledge, and says: γλυκὺ τὸ γνῶναι. But both are in reality already outside the healthy life, the *salute*, of their organism, both have lost the sweetness of pleasure and of knowledge (II, 75).

And similar to the philosophical rhetoric is the scientific rhetoric. "If philosophy has raved in its metaphysical exaltations we are putting it now on a positive ground, and here, keeping contact with reality, we have a sure road for the conquest of truth." That is what modern science says. It would be enough to ask what difference there is between reality and truth, by which, although you are in contact with reality, you must still go along a road that takes you to truth (II, 87).

Either we possess reality or we do not; either we know, and we are as many Gods in the peace of eternity; or we do not know, and—"But reality, the scientist would answer, is reality and thought is thought. When one puts his teeth in contact with an apple he needs must labor with his jaws in order to eat it. So it is with reality. At each instant of his life man comes into contact with a portion of reality; each man in his life has come into contact with one portion of reality only. Each age, each generation, each century, each civilization comes into contact with one portion only. Thousands of years shall pass, and it will never be all. What does 'either we know or we do not know' mean? We know one portion to-day and another to-morrow, and always new portions in each day of our life, and we be-

queath our several portions from generation to generation in order that the body of human science be constituted" (II, 88).

But to be able not only to bequeath his portion but even to keep it for himself each man must continually bind up its fragments σὺν αἰτίας λογισμῷ: he must treasure up his experience. And here again he anticipates in his particular knowledge the totality of knowledge: which is the αἰτία, which is the possible λογισμός, of the man who does not yet possess truth but must wait for it through the flight of thousands of years? "But here reason has only the function of giving logical connection to fragmentary experience; and the thing that matters is experience, objective experience" (II, 89).

But the objectivity of the scientist is still τρόπον τινά a subjectivity, as it is very different from the catastrophic objectivity of the man who sees things as they are, not because he needs them, but *per se,* of the man who is at last made one with the things, has all things in himself, is *persuaded,* knows. It is not the identity of my consciousness with the consciousness of things but the infinitesimal consciousness of the infinitesimal relation, and in that consciousness the illusion of the absence of any individual assent. Illusion, because the assent cannot be suppressed: to have an objective experience I must look at things that I do not see: because I see the things I see, through the assent of my whole self. And to look objectively at a given thing means to bring it near to the eye so that it may awaken its assent; not to the eye as an organ of my body but to the eye as such, as a system of lenses which should give to the thing its inorganic assent (II, 91-92).

To intensify this obtuse autonomous life of the senses science multiplies their power by means of scientific instruments. But this intensification is nothing but a repetition of the act of bringing the thing near to the eye, an

amplification of the same particular determination (II, 96). The scientist, whatever he does, always remains confined, at each moment of his activity, to a single relation, and all his efforts and instruments cannot do more than infinitesimally to reduce the extension of the given relation. But it is exactly by doing so, by essentially consisting in the repetition of the same small relations which not only do not exact but do not even tolerate the presence of the whole self, that science has planted its roots in the deepest weakness of man and given stable constitution to the rhetoric of knowledge. In the infinite number of things that they look at but do not see, the scientists bring the little light of their dark lantern to extract from the contemporaneity and succession of a given series of relations a presumption of causality: a humble hypothesis which should become a theory or a law (II, 100).

Michelstaedter is well aware that science is conscious of its finite and relative value in contrast to the infinity of its task—and that in this consciousness the scientist finds a guarantee of his own honesty. But it is precisely against this conception that he is fighting, against the affirmation of the sufficiency of a work which at each of its points is finite, as an answer to the demand for persuasion. Every particular truth of science, every portion of science, suffers from the infinite correlation with the whole of reality that science itself declares to be outside its power (II, 101, 102).

But apart from their function as researchers of truth the scientists have another which will introduce us to the last form of rhetoric, the rhetoric of society. By reproducing and simplifying given relations they are able to attain practical results; and that makes them unconscious instruments in the development of the κοινωνία κακῶν, the society of the evil ones (II, 105). Types of the κοινωνία κακῶν in different forms are Hegel's concept of state and John Stuart Mill's idea of liberty; the state that feigns to

us a larger self and an external aim to life; the liberty which consists only in the freedom of being in society, of being slaves (II, 109-111). The foundation of society is the need of securing in the future the affirmation of our own determination against all other stranger or hostile determinations (forces): of conquering matter, that is time and the variety of things or space, with our form. In matter are comprised also other men who differ from the rest of matter only in so far as they determine themselves in the same way as we do in order to continue ourselves, and impose on matter the same form that we impose on it. Our security therefore means (1) violence against nature, work; (2) violence against man, property (II, 115-116). Work and property constitute the society of man, the relation between the strong who affirms himself and the weak who sells himself, the master and the slave, both bound by the same chain to their different positions. But in a highly organized society every man imposes his violence on every other man through the omnipotence of organization; every man is matter and form, master and slave at the same time, as the common advantage grants to all the same rights and imposes on all the same duties. Through security and specialization man becomes weaker and weaker and his self more and more limited until he is little more than an inorganic will to live and everything he does is alien to him, imposed on him from the outside, not his life but his work, which he gives to society as the price of that security which otherwise could be reached only through individual superiority. And society, besides granting him the continuation of his life, the satisfaction of his inorganic will, gives to him in exchange for his work the fruition of all that human intellect has produced and accumulated in the course of centuries. An inferior individuality can thus secure for itself the fruits of the work of

superior individualities: this is the meaning of the rhetoric of society, or social optimism (II, 143, 144).

The impulse to this movement, through which the weak enjoys what rightly belongs to the strong, is given by the strong, who either through ambition or through love lay the foundations of human society. But the dream of the brotherhood of the good, ἀγαθῶν φιλία, which was in the mind and in the heart of the prophet, is the source of strength for the organization of the hostile wills that use his uncomprehended symbolical forms, the fruit of his negation, toward the security of their affirmation of life: the society of the evil ones (II,152). And they call injustice, justice; slavery, freedom; what is good for their life, morality.

The perpetuation of the social system is secured through the violence exercised on the children under the mask of love and education: what Michelstaedter calls δυσπαιδαγωγία. The cry for true life is thirst for knowledge; the great expectation, the will for good, is flattered by the fiction of a value in the social self, which is always kept before their eyes as the one that they must by imitation educate in themselves. By the system of punishments and rewards the child acquires the habit of considering his study as a necessary work if he wants to live happy, even if it is *per se* entirely alien to his life. So are imposed on him the given words, the given commonplaces, the given judgments, all the καλλωπίσματα of science and convenience which he is to take with himself to his grave. And the whole of his life will be organized on the same plan as his school-days, as the whole of human life in society is a perpetual being under age ruled by the rhetoric of duty and pleasure.

IV.

These are the main lines of Michelstaedter's critique of life, of his negation. His affirmation is in what he calls persuasion.

In whichever way man asks to continue himself, as he affirms to be just what is just for him he denies what is just for others and is unjust against all others: the affirmation of his self is always irrational and violent. But Justice (the just man, the individual who has reason in himself) is a hyperbole: that is what all say, and then turn back to live as if they had it. Hyperbolic indeed is the road of persuasion that leads to it. As the hyperbola gets infinitely nearer and nearer to the asymptote, so the man who, living, wills to live his own life, approaches to the straight line of justice; and as, however small the distance of a given point of the hyperbola from the asymptote, the curve must be infinitely prolonged to touch the straight line, so however little man, while he lives, may ask as just for himself, his duty toward justice remains infinite. The right to live cannot be paid with a finite work but only with an infinite activity. As you take part in the violence of all things all this violence is in your debt toward justice. To take it up by its roots the whole of your activity must go. To give all, to ask for nothing, such is duty (II, 45, 47).

But to give is to do the impossible: to give is to have. As long as man lives he is here, and there is the world; as long as he lives he wills to possess it; as long as he lives, in some way or other he affirms himself; he gives and asks, he enters into the cycle of relations;—and always here is he and there is the world, different from him. But in face of what was to him a given relation in which he affirmed himself, asking to continue himself, now he must affirm himself in order not to continue himself. He must love it,

not because it is necessary to his need but just for what it is in itself, he must give all to it in order to have it all. He must not see in it a particular relation but the whole world, and in relation to it he must not be his hunger, his languor, his craving for love, his need, he must be all himself. In that last present he must have all and give all, be persuaded and persuade, possess himself in the world, be one with the world. He must feel himself in the desert among the particular relations, as in none of them he can affirm himself as a whole; but in each thing that these relations offer to him he must love the life of the thing and use not the relation: affirm himself without asking. And again his life is not what this thing believes to be just for itself, he must not ask even this of the things and make of himself an instrument for their demand; for, being just to one thing, he would be unjust to the other, he would reflect the contingency of their consciousness; but he must himself will them, create them, love all himself in them, and by communicating the individual value identify himself with them (II, 49, 50).

Then the dumb and blind pain of all things, which, in so far as they will to be, are not, will have through him, who shall have taken their person, the eloquent word and the distant sight. He shall see that it is not hunger, nor thirst, nor sickness, nor misfortune, which makes man suffer; not food or drink, or apparent health, or the presence of what is in his hands, and is not his because he has no power over it, which makes him content; but he will see that in him suffers the dull pain, in each present always equally empty whether in abundance or in privation. He will suffer in the same point from his deficiency and from theirs, and, using the voice of his pain, he will speak to them with the voice of their own pain, distant itself from them. As through his intense activity he will be near to satiate his own pain, so he will bring nearer to them a life through which they will see the woof of what

oppresses them, of what successively distracts them, dissolve; will find that they are stable, without fear of instability; will suddenly perceive the walls of the little room of their misery being burst open and their little light becoming pale, in the moment when darkness will no more be outside to oppress them with its terrors,—but it will appear to them as the dawn of a new day (II, 52-53).

The man who is on the road to persuasion maintains in each point the equilibrium of his self. He does not struggle, he has no uncertainty nor weariness if he does not fear pain and has honestly identified himself with it. He lives it at every point. And as this pain is common to all things, things live in him not as the correlate of a few relations but with vastness and depth of relations. Where for others is darkness, for him is light, because the circle of his horizon is wider; where for others is misery and impotence, he has power and sees clearly. Because he has the honesty always to feel himself insufficient in face of the *infinita potestas,* he always makes himself more sufficient to things, always suffices more deeply to the eternal deficiency of things (II, 54, 55).

Therefore in his presence, in his acts, in his words, a life that transcends the shortsightedness of men reveals itself, unfolds itself, grows nearer and tangible. Hence Christ wears a halo, the stones become bread, the sick are healed, the cowards are made martyrs, and men cry: "Behold a miracle." Therefore each word of his is luminous, for they are so closely bound in their depth as to create the presence of what is distant. He can give the distant things in the near appearances, so that even he who lives only by these feels therein a sense which he was ignorant of—he can move every heart (II, 55).

The thing which he knows, and which is the taste of his wider life, is his pleasure, actual for him in every present. Alone in the desert he lives his life in dazzling vastness

THE PHILOSOPHY OF MR. B*RTR*ND R*SS*LL.

[Some further fragments found in a *Prayer Book of Free Man's Worship*[1] rescued with a few of the late Mr. R*ss*ll's belongings; see *The Monist*, Vol. XXI, October, 1911, pp. 483-508. The abbreviations used in the present instalment are as follows:

A. A. W. Lewis Carroll, *Alice's Adventures in Wonderland*, London, Macmillan, 1905. People's edition.

T. L. G. Lewis Carroll, *Through the Looking-Glass, and What Alice Found There*, London, Macmillan, 1915. People's edition.

H. S. Lewis Carroll, *The Hunting of the Snark: an Agony in Eight Fits*, London, 1911.

E. N. Richard Dedekind, *Essays on the Theory of Numbers*, Chicago and London, 1901.

P. E. Bertrand Russell, *Philosophical Essays*, London and New York, 1910.

Pr. M. Bertrand Russell, *The Principles of Mathematics*, Vol. I, Cambridge, University Press, 1903.

P. M. Alfred North Whitehead and Bertrand Russell, *Principia Mathematica*, Vol. I, Cambridge, 1910.

P. P. Bertrand Russell, *The Problems of Philosophy*, London and New York, 1912.

M. The Monist: a Quarterly Magazine Devoted to Science and Philosophy, Chicago and London.]

THE TERM "LAWS OF THOUGHT."

People often assume that laws of logic are laws of thought.[2] Perhaps the most frequent instance is the treatment of an identity as if its validity were a matter of our permission. Some people suggest to others that they should "let bygones be bygones."

[1] This apparently refers to the Essay on "The Free Man's Worship" on pp. 59-70 of the *Philosophical Essays* (London, 1910) of Mr. B*rtr*nd R*ss*ll's distinguished contemporary, Mr. Bertrand Russell, from whom Mr. R*ss*ll's philosophy was derived.

[2] Cf. *P. P.*, pp. 113, 136.

OBJECTIVE VALIDITY OF THE LAWS OF THOUGHT.

I once inquired of a maid-servant whether her mistress was at home. She replied, in a doubtful fashion, that she *thought* that her mistress was in unless she was out. I concluded that the maid was uncertain as to the objective validity of the law of the excluded middle, and remarked this to her mistress. Her mistress seemed to imagine that I wished to impute to the maid some moral defect of an unimportant nature, and remonstrated with me in an amused way. The mistress probably imagined that I meant to find fault with the maid's capacity for thinking, as I used the phrase "law of thought," and perhaps committed the common mistake of supposing that a "law of thought" has something to do with thinking.

CRITICISM.

Those people who think that it is more godlike to seem to turn water into wine than to seem to turn wine into water surprise me. I cannot imagine an intolerable critic. It seems to me that, if A resents B's criticism in trying to put his (A's) discovery in the right or wrong place, A acts as if he thought he had some private property in truth. The White Queen seems to have shared the popular misconception as to the nature of criticism.[3]

THE PRAGMATIST THEORY OF TRUTH.

The pragmatist theory that "truth" is a belief which works well sometimes conflicts with common sense and not with logic. It is commonly supposed that it is always better to be sometimes right than to be never right. But this is by no means true. For example, consider the case of a watch which has stopped; it is exactly right twice every day. A watch, on the other hand, which is always five minutes slow is never exactly right. And yet there can be

[3] See Appendix A, below.

no question but that a belief in the accuracy of the watch which was never right would, on the whole, produce better results than such a belief in the one which had altogether stopped. The pragmatist would, then, conclude that the watch which was always inaccurate gave truer results than the one which was sometimes accurate. In this conclusion the pragmatist would seem to be correct, and this is an instance of how the false premises of pragmatism may give rise to true conclusions.

From the text written above the church clock in a certain English village: "Be ye ready, for ye know not the time," we would conclude that the clock never stopped for a period as long as twelve hours. For the text is a rather vague symbolical expression of a propositional function which is asserted to be true at all instants. The proposition that a (presumably intelligent) observer of the clock at any definite instant does not know the time, implies, then, that the clock is always wrong. Now if the clock stopped for twelve hours it would be absolutely right at least once. It might be right twice if it were right at the first instant it stopped or the last instant at which it went;[4] but the second possibility is excluded by hypothesis, and the occurrence of the first possibility—or of the analogous possibility of the stopped clock being right three times in twenty-four hours—does not affect the present question. Hence the clock can never stop for twelve hours.

The pragmatist's criterion of truth appears to be far more difficult to apply than the Bellman's[5] that what he said three times is true, and to give results just as insecure.

THE SYNTHETIC NATURE OF DEDUCTION.

Doubt has often been expressed as to whether a syllogism can add to our knowledge in any way. J. S. Mill and

[4] Both cases cannot occur; the question is similar to that arising in a discussion of what is meant by saying "Socrates is mortal," see below.

[5] See Appendix B, below.

H. Poincaré in particular held the opinion that the conclusion of a syllogism is an "analytic" judgment in the sense of Kant, and therefore could be obtained by the mere dissection of the premises. Any one, then, who maintains that mathematics is founded solely on logical principles, would appear to maintain that mathematics, in the last instance, reduces to a huge tautology.

John Stuart Mill, in Chapter III of Book II of his *System of Logic,* said that "it must be granted that in every syllogism, considered as an argument to prove the conclusion, there is a *petitio principii.* When we say

All men are mortal,
Socrates is a man,
therefore
Socrates is mortal,

it is unanswerably urged by the adversaries of the syllogistic theory, that the proposition, Socrates is mortal, is presupposed in the more general assumption, All men are mortal; that we cannot be assured of the mortality of all men unless we are already certain of the mortality of every individual man; that if it be still doubtful whether Socrates, or any other individual we choose to name, be mortal or not, the same degree of uncertainty must hang over the assertion, All men are mortal; that the general principle, instead of being given as evidence of the particular case, cannot itself be taken for true without exception until every shadow of doubt which could affect any case comprised with it is dispelled by evidence *aliunde*; and then what remains for the syllogism to prove? That, in short, no reasoning from general to particular can, as such, prove anything, since from a general principle we cannot infer any particulars but those which the principle itself assumes as known. This doctrine appears to me irrefragable. . . ."

"It is," says Mr. Russell,[6] "an old debate among philosophers whether deduction ever gives *new* knowledge. We can now see that in certain cases at least it does do so. If we already know that two and two always make four, and that Brown and Jones are two, and so are Robinson and Smith, we can deduce that Brown and Jones and Robinson and Smith are four. This is new knowledge, not contained in our premises, because the general proposition, 'two and two are four,' never told us there were such people as Brown and Jones and Robinson and Smith, and the particular premises did not tell us that there were four of them, whereas the particular proposition deduced does tell us both these things. But the newness of the knowledge is much less certain if we take the stock instance of deduction that is always given in books on logic, namely 'All men are mortal; Socrates is a man, therefore Socrates is mortal.' In this case what we really know beyond reasonable doubt is that certain men, A, B, C, were mortal, since, in fact, they have died. If Socrates is one of these men it is foolish to go the roundabout way through 'all men are mortal' to arrive at the conclusion that *probably* Socrates is mortal. If Socrates is not one of the men on whom our induction is based we shall still do better to argue straight from our A, B, C, to Socrates, than to go round by the general proposition, 'all men are mortal.' For the probability that Socrates is mortal is greater, on our data, than the probability that all men are mortal. (This is obvious, because if all men are mortal, so is Socrates; but if Socrates is mortal, it does not follow that all men are mortal.) Hence we shall reach the conclusion that Socrates is mortal, with a greater approach to certainty if we make our argument purely inductive than if we go by way of 'all men are mortal' and then use deduction."

Many years ago there appeared, principally owing to

[6] *P. P.*, pp. 123-125.

the initiative of Dr. F. C. S. Schiller of Oxford, England, a comic number of *Mind*. The idea was extraordinarily good, not so the execution. A German friend of Dr. Schiller was puzzled by the appearance of the advertisements which had a doubtfully humorous appearance. However, by a syllogistic process, he acquired information which was new and useful to him, and thus incidentally refuted Mill. Presumably he started from the title of the magazine (*Mind*!), for a mark of exclamation seems nearly always in German to be a sign of an intended joke (including of course the mark after the politeness expressed in the first sentence of a private letter or a public address). There would be, then, the following syllogism:

> This is a book of would-be jokes (i. e., everything in this book is a would-be joke);
> This advertisement is in this book;
> Therefore, this advertisement is a would-be joke.

Thus the syllogism may be almost as powerful an agent in the detection of humor as M. Bergson's criterion shortly to be described.

THE MORTALITY OF SOCRATES.

The mortality of Socrates is so often asserted in books on logic that it may be as well briefly to consider what it means. The phrase "Socrates is mortal" may be thus defined: "There is at least one instant t such that t has not to Socrates the one-many relation R which is the converse of the relation 'exist at,' and all instants following t have not the relation R to Socrates, and there is at least one instant t' such that neither t' nor any instant preceding t' has the relation R to Socrates."

This definition has many merits. In the first place, no assumption is made that Socrates ever lived at all. In the second place, no assumption is made that the instants of time form a continuous series. In the third place, no as-

sumption is made as to whether Socrates had a first or last moment of his existence. If time be indeed a continuous series, then we can easily deduce[7] that there must have been *either* a first moment of his existence *or* a last one, but not both; just as there seems to be either a greatest weight that a man can lift or a least weight that he cannot lift, but not both.[8]

IMPLICATION.

A distinguished philosopher (M) once thought that the logical use of the word "implication,"—any false proposition being said to "imply" any proposition true or false,—is absurd, on the grounds that it is ridiculous to suppose that the proposition "2 and 2 make 5" implies the proposition "M is the Pope." This is a most unfortunate instance, because it so happens that the false proposition that 2 and 2 make 5 can rigorously be proved to imply that M, or anybody else other than the Pope, is the Pope. For if 2 and 2 make 5, since they also make 4, we could conclude that 5 is equal to 4. Consequently, subtracting 3 from both sides, we conclude that 2 would be equal to 1. But if this were true, since M and the Pope are two, they would be one, and obviously then M would be the Pope.

DENIAL OF GENERALITY, AND GENERALITY OF DENIAL.

The conclusion of a certain song[9] about a young man who poisoned his sweetheart with sheep's-head broth, and was frightened to death by a voice exclaiming:

> "Where's that young maid
> What you did poison with my head?"

at his bedside, gives rise to difficulties which are readily solved by a symbolism that brings into relief the principle that the denial of a universal and non-existential proposition

[7] From "Dedekind's axiom" (*E. N.*, p. 11).

[8] Cf. *M.*, April, 1908.

[9] To which De Morgan drew attention in a letter; see (Mrs.) S. E. De Morgan, *Memoir of Augustus De Morgan*, London, 1882, p. 324.

is a particular and existential one. The conclusion of the song is:

> "Now all young men, both high and low,
> Take warning by this dismal go!
> For if he'd never done nobody no wrong,
> He might have been here to have heard the song."

It is an obvious error, say Whitehead and Russell,[10] though one easy to commit, to assume that the cases: (1) all the propositions of a certain class are true; and (2) no proposition of the class is true; are each other's contradictories. However, in the modification[11] of Frege's symbolism which was used by Russell,

(1) is $(x).\ x,$

and (2) is $(x).\ \text{not}\ x;$

while the contradictory of (1) is

$$\text{not}\ (x).\ x.$$

The last line of the above verse may, then, be written

$$(t).\text{not}\ (x).\ \text{not not}\,\varphi(x,t),$$

where "$\varphi(x,t)$" denotes the unasserted propositional function "the doing wrong to the person x at the instant t." By means of the principle of double negation we can at once simplify the above expression into:

$$(t).\text{not}\,(x).\varphi(x,t);$$

which can be thus read: "If at every instant of his life there was at least one person x to whom he did no wrong (at that instant)." It is difficult to imagine any one so sunk in iniquity that he would not satisfy this hypothesis. We are forced, then, unless our imagination for evil is to be distrusted, to conclude that any one might have been there to have heard that song. Now this conclusion is probably

[10] *P. M.*, p. 16.

[11] However, here, for the printer's convenience, we depart from Mr. Russell's usage so far as to write "not" for a curly minus-sign.

false, possibly on physical grounds, and certainly on esthetic grounds.

According as the symbol for "not" comes before the (x) or between the (x) and the φx, we have an expression of what Frege called respectively the denial of generality, and the generality of denial. The denial of the generality of a denial is the form of all existential propositions, while the assertion of or denial of generality is the general form of all non-existential or universal propositions.

LOGICAL ADDITION AND THE NATURE OF SYMBOLISM.

Frequently ordinary language contains subtle psychological implications which cannot be translated into symbolic logic except at great length. Thus if a man (say Mr. Jones) wishes to speak collectively of himself and his wife, the order of the mentioning of the terms in the class considered and the names applied to these terms are, logically speaking, irrelevant. And yet, more or less definite information is given about Mr. Jones, provided that he is an Englishman, according as he talks to his friends of

(1) Mrs. Jones and I,
(2) I (or me) and my wife (or missus),
(3) My wife and I,
or (4) I (or me) and Mrs. Jones.

In case (1) one is probably correct in placing Mr. Jones among the clergy or the small professional men who make up the bulk of the middle-class; in case (2) one would conclude that Mr. Jones belonged to the lower middle-class; the form (3) would be used by Mr. Jones if he were a member of the upper, upper middle, or lower classes; while form (4) is only used by retired shop-keepers of the lower mi[illegible] of which a male member usually combines [illegible]premacy of man with belief in the dignity of [illegible] himself. A further complication is

introduced if a wife is referred to as "the wife." This has already been briefly referred to in my note on "The."[12] Cases (2) and (3) then each give rise to one more case. Cases (1) and (4) do not, since nobody has hitherto referred to his wife as "the Mrs. Jones,"—at least without a qualifying adjective before the "Mrs."

On the other hand, certain descriptive phrases and certain propositions can be expressed more shortly and more accurately by means of symbolic logic. Let us consider the proposition: "No man marries his deceased wife's sister." If we assume, as a first approximation, that all marriages are fertile, and that all children are legitimate, then, with only four primitive ideas: the relation of parent to child (P) and the three classes of males, females, and dead people, we can define "wife" (a female who has the relation formed by taking the relative product of P and $\breve{P}$[13] to a male), "sister," "deceased wife," and "deceased wife's sister" in terms of these ideas and of the fundamental notions of logic. Then the proposition: "no man marries his deceased wife's sister," can be expressed unambiguously by about twenty-nine simple signs on paper, whereas, in words, the unasserted statement consists of no less than thirty-four letters. Although, legally speaking, we should have to adopt somewhat different definitions and possibly increase the complications of our proposition, it must be remembered that, on the other hand, we always reduce the number of symbols in any proposition by increasing the number of definitions in the preliminaries to it.

By such means we may advance a step toward making legal definitions and propositions exact, and thus logic can make contributions to law in return for those made by law to logic.[14]

[12] See *M.*, Oct., 1911, Vol. XXI, p. 492.
[13] C. S. Peirce's notation for the relation "converse of P."
[14] See *M*, Oct., 1911, Vol. XXI, pp. 484-485, 492.

IDENTITY OF CLASSES.

I once heard of a somewhat meritorious lady who was extremely conventional, and, on the slender grounds of carefully acquired habits of preferring the word "woman" to the word "lady" and of going to the post-office without a hat, imagined that she was unconventional and altogether a remarkable person, and once remarked with great satisfaction that she was a very queer person, and that nothing shocked her "except, of course, bad form."

Thus, she asserted that all the things which shocked her were actions in bad form; and she would undoubtedly agree, though she did not actually state it, that all the things which were done in bad form would shock her. Consequently she asserted that the class of things which shocked her was the class of actions in bad form. Consequently the statement of this lady that some or all of the actions done in bad form shocked her is an identical proposition of the form: "nothing shocks me, except, of course, the things which do, in fact, shock me"; and this statement the lady certainly did not intend to make.

This excellent lady, had she but known it, was logically justified in making any statement whatever about her unconventionality. For the class of her unconventional actions was the null class. Thus she might logically have made inconsistent statements about this class of actions. As a matter of fact she did make inconsistent statements, but unfortunately she justified them by stating that, "It is the privilege of woman to be inconsistent." She was one of those persons who say things like that.

ETHICAL APPLICATIONS OF THE LAW OF IDENTITY.

It may be remembered that Mr. Podsnap remarked, with sadness tempered by satisfaction, that he regretted to say that "Foreign nations do as they do do." Besides aiding the comforting expression of moral disapproval, the

law of identity has yet another useful purpose in practical ethics: It serves the welcome purpose of providing an excuse for infractions of the moral law. There was once a man who treated his wife badly, was unfaithful to her, was dishonest in business, and was not particular in his use of language; and yet his life on earth was described in the lines:

> "This man maintained a wife's a wife,
> Men are as they are made,
> Business is business, life is life,
> And called a spade a spade."

One of the objects of Mr. G. E. Moore's *Principia Ethica*[15] was to argue that the word "good" means simply good, and not pleasant or anything else. Appropriately enough, this book bore on its title-page the quotation from the preface to the *Sermons,* published in 1726, of Bishop Joseph Butler, the author of the *Analogy*: "Everything is what it is and not another thing."

But another famous Butler—Samuel Butler, the author of *Hudibras,*—went farther than this and maintained that identities were the highest attainment of metaphysics itself. At the beginning of the first Canto of *Hudibras,* in the description of Hudibras himself, Butler wrote:

> "He knew what's what, and that's as high
> As metaphysic wit can fly."

I once conducted what I imagined to be an esthetic investigation for the purpose of discovery, by the continual use of the word "Why?"[16] the grounds upon which certain people choose to put milk into a tea-cup before the tea. I was surprised to discover that it was an ethical, and not an esthetic problem; for I soon elicited the fact that it was done because it was "right." A continuance of my patient questioning elicited further evidence of the fundamental character of the principle of identity in ethics; for it was right, I learned, because "right is right."

[15] Cambridge, 1903.

[16] Cf. *P. E.*, p. 2.

It appears that some people unconsciously think that the principle of identity is the foundation, in certain religions, of the reasons which can be alleged for moral conduct, and are surprised when this fact is pointed out to them. The late Sir Leslie Stephen, when traveling by railway, fell into conversation with an officer of the Salvation Army, who tried hard to convert him. Failing in this laudable endeavor, the Salvationist at last remarked: "But if you aren't saved, you can't go to heaven!" "That, my friend," replied Stephen, "is an identical proposition."

DIGNITY.

We have seen[17] that logical implication is often an enemy of dignity. The subject of dignity is not usually considered in treatises on logic, but, as we have remarked,[18] many mathematicians implicitly or explicitly seem to fear either that the dignity of mathematics will be impaired if she follow out conclusions logically, or that only an act of faith can save us from the belief that, if we followed out conclusions logically, we should find out something alarming about the past, present, or future of mathematics.

Thus it seems necessary to inquire rather more closely into the nature of dignity, with a view to the discovery of whether it is, as is commonly supposed, a merit in life and logic.

The chief use of dignity is to veil ignorance. Thus it is well known that the most dignified people, as a rule, are schoolmasters; and schoolmasters are usually so occupied with teaching that they have no time to learn anything. And because dignity is used to hide ignorance, it is plain that impudence is not always the opposite of dignity, but that dignity is sometimes impudence. Dignity is said to inspire respect; and this may be in part why respect for

[17] *M*, Oct., 1911, Vol. XXI, p. 497.
[18] *Ibid.* Cf. also the section below on "The Paradoxes of Logic."

others is an error of judgment and self-respect is ridiculous.

Self-respect is, of course, self-esteem. William James has remarked that self-esteem depends, not simply upon our success, but upon the ratio of our success to our pretensions, and can therefore be increased by diminishing our pretensions. Thus if a man is successful, but only then, can he be both ambitious and dignified. James also implies that happiness increases with self-esteem. Likeness of thought with one's friends, then, does not make one happy, for otherwise a man who esteemed himself little would be indeed happy. Also if a man is unhappy he could not, from our premises, by the principles of the syllogism and of contraposition, be dignified,—a conclusion which should be fatal to many novelists' heroes.

A reflection on pessimism to which this discussion gives rise is the following. It would appear that a man's self-esteem would be increased by a conviction of the unworthiness of his neighbors. A man, therefore, who thinks that the world and all its inhabitants, except himself, are very bad, should be extremely happy. In fact the effects would hardly be distinguishable from those of optimism. And optimism, as everybody knows, is a state of mind induced by stupidity.

THE PARADOXES OF LOGIC.

We have already[19] referred to the contempt shown by some mathematicians for exact thought, which they condemn under the name of "scholasticism." An example of this is given by Schoenflies in the second part of his publication usually known as the *Bericht über Mengenlehre.*[20] Here[21] a battle-cry in italics:

[19] *M*, Oct., 1911, Vol. XXI, p. 486.

[20] *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten. Bericht, erstattet der deutschen Mathematiker-Vereinigung*, Leipsic, 1908.

[21] *Ibid.*, p. 7. The battle-cry is: *"Gegen jede Resignation, aber auch gegen jede Scholastik!"*

"Against all resignation, but also against all scholasticism!"

found utterance. Later on Schoenflies[22] got bolder and adopted a more personal battle-cry, also in italics and with a whole line to itself:

"For Cantorism but against Russellism!"

"Cantorism" means the theory of transfinite aggregates and numbers erected for the most part by Georg Cantor. Shortly speaking, the great sin of "Russellism" is to have gone too far in the chain of logical deduction for many mathematicians, who were perhaps, like Schoenflies,[23] blinded by their rather uncritical love of mathematics. Thus it comes about that Schoenflies[24] denounces Russellism as "scholastic and unhealthy." This queer blend of qualities would surely arouse the curiosity of the most *blasé* as to what strange thing Russellism must be.[25]

Schoenflies[26] said that some mathematicians attributed to the logical paradoxes which have given Russell so much trouble to clear up, "especially to those that are artificially constructed, a signification that they do not have." Yet no grounds were given for this assertion, from which it might be concluded that the rigid examination of any concept was unimportant. The paradoxes are simply the necessary results of certain logical views which are currently held, which views do not, except when they are examined rather closely, appear to contain any difficulty. The contradiction is not felt, as it happens, by people who confine

[22] "Ueber die Stellung der Definition in der Axiomatik," *Jahresber. der deutsch. Math.-Ver.*, Vol. XX, 1911, pp. 222-255. The battle-cry is on p. 256 and is: *"Für den Cantorismus aber gegen den Russellismus!"*

[23] *Ibid.*, p. 251. *"Es ist also,"* he exclaims with the eloquence of emotion and the emotion of eloquence, *"nicht die Geringschätzung der Philosophie, die mich dabei treibt, sondern die Liebe zur Mathematik;...."*

[24] *Ibid.*

[25] Cf. for this, *M*, Jan., 1912, Vol. XXII, pp. 149-158.

[26] *Bericht*, 1908, p. 76*n*; cf. p. 72.

their attention to the first few number-classes of Cantor, and this seems to have given rise to the opinion, which it is a little surprising to find that some still hold, that cases not usually met with, though falling under the same concept as those usually met with, are of little importance. One might just as well maintain that continuous but not differentiable functions are unimportant because they are artificially constructed,—a term which I suppose means that they do not present themselves when unasked for. Rather should we say that it is by the discovery and investigation of such cases that the concept in question can alone be judged, and the validity of certain theorems—if they are valid—conclusively proved. That this has been done, chiefly by the work of Russell, is simply a fact; that this work has been and is misunderstood by many[27] is regrettable for this reason, among others, that it proves that, at the present time, as in the days in which *Gulliver's Travels* were written, some mathematicians are bad reasoners.

Nearly all mathematicians agreed that the way to solve these paradoxes was simply not to mention them; but there was some divergence of opinion as to how they were to be unmentioned. It was clearly unsatisfactory merely not to mention them. Thus Poincaré was apparently of opinion that the best way of avoiding such awkward subjects was to mention that they were not to be mentioned. But[28] "one might as well, in talking to a man with a long nose, say: 'When I speak of noses, I except such as are inordinately long,' which would not be a very successful effort to avoid a painful topic."

Schoenflies, in his paper of 1911 mentioned above, adopted the convenient plan of referring these logical difficulties at the root of mathematics to a department of

[27] E. g., in F. Hausdorff's review of Russell's *Principles* of 1903 in the *Vierteljahrsschr. für wiss. Philos. und Soziologie.*

[28] Russell, *A. J. M.*, Vol. XXX, 1908, p. 226.

knowledge which he called "philosophy." He said[29] of the theory of aggregates that though "born of the acuteness of the mathematical spirit, it has gradually fallen into philosophical ways, and has to some extent the compelling force which dwells in the mathematical process of conclusion."

The majority of mathematicians have followed Schoenflies rather than Poincaré, and have thus adopted tactics rather like those of the March Hare and the Gryphon,[30] who promptly changed the subject when Alice raised awkward questions. Indeed, the process of the first of these creatures of a child's dream is rather preferable to that of Schoenflies. The March Hare refused to discuss the subject because he was bored when difficulties arose. Schoenflies would not say that he was bored,—he professed interest in philosophical matter,—but simply called the logical continuation of a subject by another name when he did not wish to discuss this continuation, and thus implied that he had discussed the whole subject. Further, Schoenflies would not apparently admit that the one method of logic could be applied to the solution of both mathematical and philosophical problems, in so far as these problems are soluble at all; but the March Hare, shortly before the remark we have just quoted, rightly showed great astonishment that butter did not help to cure both hunger and watches that would not go.[31] The judgment of Schoenflies by which certain apparently mathematical questions were condemned as "philosophical" rested on grounds as flimsy as those in the Dreyfus Case or the Trial in *Wonderland*.[32]

MODERN LOGIC AND SOME PHILOSOPHICAL ARGUMENTS.

The most noteworthy reformation of recent years in logic is the discovery and development by Mr. Bertrand

[29] *Loc. cit.*, p. 222.
[30] See Appendix C.
[31] See Appendix D.
[32] See Appendix E.

Russell of the fact that the paradoxes,—of Burali-Forti, Russell, König, Richard and others,—which have appeared of late years in the mathematical theory of aggregates and have just been referred to, are of an entirely *logical* nature, and that their avoidance requires us to take account of a principle which has been hitherto unrecognized, and which renders several well-known arguments in refutation of scepticism, agnosticism, and the statement of a man that he asserts nothing, invalid.

Dr. Whitehead and Mr. Russell say in their *P. M.*:[33] "The principle which enables us to avoid illegitimate totalities may be stated as follows: 'Whatever involves *all* of a collection must not be one of the collection,' or conversely: 'If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total.' We shall call this the 'vicious-circle principle,' because it enables us to avoid the vicious circles involved in the assumption of illegitimate totalities. Arguments which are condemned by the vicious-circle principle will be called 'vicious-circle fallacies.' Such arguments, in certain circumstances, may lead to contradictions, but it often happens that the conclusions to which they lead are in fact true, though the arguments are fallacious. Take, for example, the law of excluded middle in the form 'all propositions are true or false.' If from this law we argue that, because the law of excluded middle is a proposition, therefore the law of excluded middle is true or false, we incur a vicious-circle fallacy. 'All propositions' must be in some way limited before it becomes a legitimate totality, and any limitation which makes it legitimate must make any statement about the totality fall outside the totality. Similarly the imaginary sceptic who asserts that he knows nothing and is refuted by being asked if he knows that he knows nothing, has asserted nonsense, and has been falla-

[33] P. 40.

ciously refuted by an argument which involves a vicious-circle fallacy. In order that the sceptic's assertion may become significant it is necessary to place some limitation upon the things of which he is asserting his ignorance; the proposition that he is ignorant of every member of this collection must not itself be one of the collection. Hence any significant scepticism is not open to the above form of refutation."

THE HIERARCHY OF JOKES.

Jokes may be divided into various types. Thus a joke or class of jokes which is itself the subject of a joke can only be the subject of a joke of higher order. Otherwise we would get the same vicious-circle fallacy which gives rise to so many paradoxes in logic and mathematics. Thus a certain Oxford scholar succeeded, to his own satisfaction, in reducing all jokes to primitive types consisting of thirty-seven proto-Aryan jokes. When any proposition was propounded to him he would reflect and afterwards pronounce on the question as to whether the proposition was a joke or not. If he decided, by his theory, that it was a joke he would solemnly say: "There is that joke." If this narration is accepted as a joke, since it cannot be reduced to one of the proto-Aryan jokes under pain of leading us to commit a vicious-circle fallacy, we must conclude that there is at least one joke which is not proto-Aryan; and, in fact, is of a higher type. There is no great difficulty, in point of principle, in forming a hierarchy of jokes of various types. Thus a joke of the fourth type (or order) is as follows: A joke of the first order was told to a Scotchman, who, as we would expect, was unable to see it. The person (A) who told this joke told the story of how the joke was received to another Scotchman, thereby making a joke about a joke of the first order, and thus making a joke of the second order. A remarked on this joke that no joke

could penetrate the head of the Scotchman to whom the joke of the first order was told even if it were fired into his head with a gun. The Scotchman, after severe thought, replied: "But ye couldn't do that, ye know!" *A* repeated the whole story, which constituted a joke of the third order, to a third Scotchman. This Scotchman again, after prolonged thought, replied: "He had ye there!" This whole story is a joke of the fourth order.

Most known jokes are of the first order, for the simple reason that the majority of people find that the slightest mental effort effectually destroys any perception of humor. It seems to me that a joke becomes more pleasurable in proportion as logical faculties are brought into play by it, and hence that logical power is allied, or possibly identical, with the power of grasping more subtle jokes. The jokes which amuse the frequenters of music-halls, Conservatives, and Mr. Bergson—and which usually deal with accidents, physical defects, mothers-in-law, foreigners, or over-ripe cheese—are usually jokes of the first order. Jokes of the second, and even of the third, order appeal to ordinary well-educated people; jokes of higher order require either special ability or a sound logical training on the part of the hearer if the joke is to be appreciated; while jokes of transfinite order presumably only excite the inaudible laughter of the gods.

LAUGHTER.

In a review[34] of Bergson's book on *Laughter*,[35] Mr. B. Russell has remarked:

"It has long been recognized by publishers that everybody desires to be a perfect lady or gentleman (as the case may be); to this fact we owe the constant stream of

[34] "The Professor's Guide to Laughter," *The Cambridge Review*, Vol. XXXII, 1912, pp. 193-194.

[35] *Laughter, an Essay on the Meaning of the Comic*, English translation by C. Brereton and F. Rothwell, London, 1911.

etiquette-books. But if there is one thing which people desire even more, it is to have a faultless sense of humor. Yet so far as I know there is no book called 'Jokes without Tears, by Mr. McQuedy.' This extraordinary lacuna has now been filled. Those to whom laughter has hitherto been an unintelligible vagary, in which one must join, though one could never tell when it would break out, need only study Mr. Bergson's book to acquire the finest flower of Parisian wit. By observing a very simple formula they will know infallibly what is funny and what is not; if they sometimes surprise their unlearned friends they have only to mention their authority in order to silence doubt. 'The attitudes, gestures and movements of the human body,' says M. Bergson, 'are laughable in exact proportion as that body reminds us of a mere machine.' When an elderly gentleman slips on a piece of orange peel and falls, we laugh, because his body follows the laws of dynamics instead of a human purpose. When a man falls from a scoffolding and breaks his neck on the pavement, we presumably laugh even more, since the movement is even more completely mechanical. When the clown makes a bad joke for the first time, we keep our countenance, but at the fifth repetition we smile, and at the tenth we roar with laughter, because we begin to feel him a mere automaton. We laugh at Molière's misers, misanthropists and hypocrites, because they are mere types mechanically dominated by a master impulse. Presumably we laugh at Balzac's characters for the same reason; and presumably we never smile at Falstaff, because he is individual throughout."

The review concludes with the reflection that "it would seem to be impossible to find any such formula as M. Bergson seeks. Every formula treats what is living as if it were mechanical, and is therefore by his own rule a fitting object of laughter." Now this undoubtedly true conclu-

sion has been obtained, as is readily seen, by a vicious-circle fallacy.

HISTORICAL CRITICISM.

From a problem in Diophantus's *Arithmetic* about the price of some wine it would seem that the wine was of poor quality, and Paul Tannery has suggested that the prices mentioned for such a wine are higher than were usual until after the end of the second century. He therefore rejected the view which was formerly held that Diophantus lived in that century.[36]

The same method applied to a problem given by the ancient Hindu algebraist Brahmagupta, who lived in the seventh century after Christ, might result in placing Brahmagupta in prehistoric times. This is the problem:[37] "Two apes lived at the top of a cliff of height h, whose base was distant mh from a neighboring village. One descended the cliff and walked to the village, the other flew up a height x and then flew in a straight line to the village. The distance traversed by each was the same. Find x."

THE HUMOR OF MATHEMATICIANS.

Brahmagupta's problem appears to be the earliest instance of a kind of joke which has been much used by mathematicians. For the sake of giving a certain picturesqueness to the data of problems and so to excite that sort of interest which is partly expressed by a smile, mathematicians have got into the habit of talking, for example, of monkeys in the form of geometrical points climbing up massless ropes. Prof. P. Stäckel[38] truly remarked that physiological mechanics—the mechanics of bones, muscles, and so on—is wholly different from this. There was once a

[36] W. W. Rouse Ball, *A Short Account of the History of Mathematics*, 4th ed., London, 1908, p. 109.

[37] *Ibid.*, pp. 148-149.

[38] *Encykl. der math. Wiss.*, Vol. IV, part I, p. 474.

lecturer on mathematics at Cambridge, England, who used yearly to propound to his pupils a problem in rigid dynamics which related to the motion of a garden roller supposed to be without mass or friction, when a heavy and perfectly rough insect walked round the interior of it in the direction of normal rolling.

Hitherto this has been the only mathematical outlet for the humor of mathematicians; and those who really had the interests of mathematics at heart saw with alarm the growing tendency towards scholasticism in mathematical jokes. Fortunately the discovery of logic by some mathematicians has removed this danger. Still to many mathematicians logic is still unknown, and to them—to Prof. A. Schoenflies for example—modern mathematics, owing to its alliance with logic, appears to be sinking into scholasticism. It is true that the word "scholasticism" is not used by Professor Schoenflies in any intentionally precise signification, but merely as a vague epithet of disapproval, very much as the word "socialism" is used by the ordinary philistine, and this would certainly serve as a sufficient excuse. But no excuse is needed: these opinions are themselves a source of mathematical jokes.

THE CONVERSION OF RELATIONS.

The "Conversion of Relations" does not mean what it might be supposed to mean; it has nothing to do with what Kant called "the wholesome art of persuasion." What concerns us here is the convertibility of a logical relation. If A has a certain relation R to B, the relation of B to A, which may be denoted by $\breve{R}$, is called the *converse* of R. As De Morgan[39] remarked, this conversion may sometimes present difficulties. The following is De Morgan's example:

"Teacher: 'Now, boys, Shem, Ham and Japheth were

[39] *Trans. Camb. Phil. Soc.*, Vol. X, 1864, part II, note on page 334.

Noah's sons; who was the father of Shem, Ham and Japheth?' No answer.

"Teacher: 'Boys, you know Mr. Smith, the carpenter, opposite; has he any sons?'

"Boys: 'Oh! yes Sir! there's Bill and Ben.'

"Teacher: 'And who is the father of Bill and Ben Smith?'

"Boys: 'Why Mr. Smith to be sure.'

"Teacher: 'Well, then, once more, Shem, Ham and Japheth were *Noah*'s sons; who was the father of Shem, Ham and Japheth?'

"A long pause; at last a boy, indignant at what he thought the attempted trick, cried out: 'It *couldn't* have been Mr. Smith.' These boys had never converted the relation of father and son,. . ."

FINITE AND INFINITE.

I was once shown a statement made by an eminent mathematician of Cambridge (England) from which one would conclude that this mathematician thought that finite distances became infinite when they were great enough. In one of those splendidly printed books, bound in blue, published by the University Press, and sold at about a guinea as a guide to some advanced branch of pure mathematics, one may read, even in the second edition published in 1900, the words: "Representation [of a complex variable] on a plane is obviously more effective for points at a finite distance from the origin than for points at a very great distance."

Plainly some of the points at a very great distance are at a *finite* distance, for the same author mentions that Neumann's sphere for representing the positions of points on a plane "has the advantage. . . .of exhibiting the uniqueness of $z=\infty$ as a value of the variable."

THE MATHEMATICAL ATTAINMENTS OF TRISTRAM SHANDY.

Tristram Shandy[40] said that his father was sometimes a gainer by a misfortune; for if the pleasure of haranging about it was as ten, and the misfortune itself only as five, he gained "half in half," and was well off again as if the misfortune had never happened.

Suppose that the unit (arbitrary) of pleasure is denoted by A, Tristram Shandy, by neglecting, in this ethical discussion, to introduce negative quantities (Kant's pamphlet advocating this introduction into philosophy was made subsequently)[41] apparently made 15 A to result, and this can hardly be maintained to be the half of 10 A. It is possible however that Tristram Shandy succeeded in proving the apparently paradoxical equation

$$15\,A = 5A$$

by remarking that the axiom "the whole is greater than the part" does not always hold. This remark follows at once from what Mr. Russell[42] has called "The Paradox of Tristram Shandy." This paradox is described by Mr. Russell as follows:

"Tristram Shandy, as we know, took two years writing the history of the first two days of his life, and lamented that, at this rate, material would accumulate faster than he could deal with it, so that he could never come to an end. Now I maintain that, if he had lived for ever, and not wearied of his task, then, even if his life had continued as eventfully as it began, no part of his biography would have remained unwritten."

This paradox is strictly correlative to the well-known

[40] Cf. a letter of De Morgan's in Mrs. De Morgan's *Memoir of Augustus De Morgan*, p. 324.

[41] Kant's tract was published in 1763, while *Tristram Shandy* was published in 1760.

[42] *Pr. M.*, pp. 358-359; cf. *M*, Jan., 1912, Vol. XXII, p. 187.

paradox of Zeno's about Achilles and the Tortoise.[43] "The Achilles proves that two variables in a continuous series, which approach equality from the same side, cannot ever have a common limit: the Tristram Shandy proves that two variables which start from a common term, and proceed in the same direction, but diverge more and more, may yet determine the same limiting class (which however is not necessarily a segment, because segments were defined as having terms beyond them). The Achilles assumes that whole and part cannot be similar, and deduces a paradox; the other, starting from a platitude, deduces that whole and part may be similar. For common sense, it must be confessed, that is a most unfortunate state of things." And Mr. Russell considers that, in the face of proofs, it ought to commit suicide in despair.

Now I suggest the extremely unlikely possibility that Tristram Shandy, by reflection on his own life and literary labors, was led to the correct course of accepting the paradox which resulted from this reflection and rejecting the Achilles. Thus, he concluded that an infinite whole may be similar (or, in Cantor's terminology, equivalent) to a proper part of itself, and hence, by a confusion of similarity with identity (or equivalence with equality) which he shares with some subsequent philosophers,[44] that a whole may be equal to a proper part of itself. If A is an infinite class it is not difficult to see that we can have

$$10A = 5A.$$

In this way many have avoided an opinion which rests on no better foundation than that formerly entertained by the inductive philosophers of Central Africa, that all men are black.[45]

[43] Cf. *Pr. M.*, pp 350, 358-359; *M.*, Vol. XXII, 1912, p. 157.
[44] Cf., for example, Cosmo Guastella, *Dell' infinito*, Palermo, 1912.
[45] Cf. Russell, *Pr. M.*, p. 360.

THE HARDSHIPS OF A MAN WITH AN UNLIMITED INCOME.

I once heard a man refer to his income as limited, in order to illustrate the hardships of a class of men, of which he of course was one, in having to pay a somewhat high income-tax. It is obvious that this man spoke enviously, and consequently admitted the existence of more fortunately placed individuals such that at least one had an unlimited income. A little reflection would have shown the man that he was not taking up a paradoxical attitude. A "paradoxical attitude" is of course the assertion of one or more propositions of which the truth cannot be perceived by a philosopher—and particularly an idealist—and can be perceived by a logician and occasionally but not always by a man of common sense. Such propositions are: "The cat is hungry," "Columbus discovered America," and "A thing which is always at rest may move from the position A to the different position B."

Now if a man had an unlimited income it is an immediate inference that, however low income-tax might be, he would have to pay annually to the exchequer of his nation a sum equal in value to his whole income. Further, if his income was derived from a capital invested at a finite rate of interest (as is usual), the annual payments of income-tax would each be equal in value to the man's whole capital. If, then, the man with an unlimited income chose to be discontented, he would be sure of a sympathetic audience among philosophers and business acquaintances; but discontent could not last long, for the thought of the difficulties he was putting in the way of the chancellor of the exchequer, who would find the drawing up of his budget most puzzling, would be amusing. Again, the discovery that, after paying an infinite income-tax, the income would be quite undiminished, would obviously afford an uneasy satisfaction.

THE RELATIONS OF MAGNITUDE OF CARDINAL NUMBERS.

The theorems of cardinal arithmetic are frequently used in ordinary conversation. What is known as the Schröder-Bernstein theorem was used, long before Bernstein or Schröder, by Thurlow, afterward the law-lord Lord Thurlow, when an undergraduate of Trinity College, Dublin. Thurlow was rebuked for idleness by the provost (I think) who said to him: "Whenever I look out of the window, Mr. Thurlow, I see you crossing the court." The provost thus asserted a one-one correspondence between the class A of his acts of looking out of the window and a part of the class B of Thurlow's acts of crossing the court. Thurlow asserted in reply a one-one correspondence between B and a part of A: "Whenever I cross the court I see you looking out of the window." The Schröder-Bernstein theorem then allows us to conclude that there is a one-one correspondence between the classes A and B. That A and B were finite classes is not the fault of the provost or Thurlow; nor is it relevant logically.

THE EVIDENCE OF GEOMETRICAL PROPOSITIONS.

It has often been maintained that the twentieth proposition of the first book of Euclid—that two sides of a triangle are together greater than the third side—is evident even to asses. This does not however seem to me generally true. I once asked a coastguardsman the distance from A to B; he replied: "eight miles." On further inquiry I elicited the fact that the distance from A to C was two miles and the distance from C to B was twenty-two miles. Now the paths from A to B and from C to B were by sea, while the path from A to C was by land. Hence if the path by land was rugged and the distance along the road was two miles, it would appear that the coastguardsman believed that not only could one side of a triangle be

board geometry, a wooden geometry, a stone geometry, and so on;[47] but also that the foundations of logic are inductive in their nature.[48] "We cannot," says Mill,[49] conceive a round square, nor merely because no such object has ever presented itself in our experience, for that would not be enough. Neither, for anything we know, are the two ideas in themselves incompatible. To conceive a body all black and yet white, would only be to conceive two different sensations as produced in us simultaneously by the same object—a conception familiar to our experience—and we should probably be as well able to conceive a round square as a hard square, or a heavy square, if it were not that in our uniform experience, at the instant when a thing begins to be round, it ceases to be square, so that the beginning of the one impression is inseparably associated with the departure or cessation of the other. Thus our inability to form a conception always arises from our being compelled to form another contradictory to it."

THE LAW OF CONTRADICTION.

Considering the important place assigned by philosophers and logicians to the law of contradiction, the remark will naturally be resented by many of the older schools of philosophy and especially by Kantians, that "in spite of its fame we have found few occasions for its use."[50]

THE PRINCIPLE OF PERMANENCE.

In their readiness to consider many different things as one thing,—to consider, for example the ratio 2:1 as the same thing as the cardinal number 2,—such mathematicians as Peacock, Hankel and Schubert were forestalled

[47] J. B. Stallo, *The Concepts and Theories of Modern Physics*, 4th ed., London, 1900, pp. 217-227.

[48] *Ibid.*, pp. 140-144.

[49] *Examination of the Philosophy of Sir William Hamilton*, Vol. I, p. 88, Amer. ed.

[50] *P. M.*, p. 116.

by the Pigeon, who thought that Alice and the Serpent were the same creature, because both had long necks and ate eggs.[51]

It is however doubtful whether the Pigeon would have followed the example of the mathematicians just mentioned so far as to embrace the creed of nominalism and so to feel no difficulty in subtracting from zero,—a difficulty which was pointed out with great acuteness by the Hatter[52] and modern mathematical logicians.

NOMINALISM.

One of the chief differences between logicians and men of letters is that the latter mean many different things by one word, whereas the former do not—at least nowadays. Most mathematicians belong to the class of men of letters.

I once had a manservant who told me on a certain occasion that he "never thought a word about it." I was doubtful whether to class him with such eminent mathematicians as Helmholtz, Kronecker, Stolz, Pringsheim and Schubert, or as a supporter of Max Müller's theory of the identity of thought and language. However since the man was very untruthful, and I have heard that he meant what he said and said what he meant,[53] the conclusion is probably correct that he really believed that the meanings of his words were not the words themselves. Thus I think it most probable that my manservant had been a mathematician but had escaped by the aid of logic.

IS THE MIND IN THE HEAD?

The contrary opinion has been maintained by idealists and a certain election agent with whom I once had to deal,

[51] See Appendix F.

[52] See Appendix G.

[53] The Hatter (see Appendix H) pointed out that there is a difference between these two assertions. Thus he clearly showed that he was a nominalist, and philosophically opposed to the March Hare, who had recommended Alice to say what she meant.

and who remarked that something slipped his mind and then went out of his head altogether. At some period, then, a remembrance was in his head and out of his mind; his mind was not, then, wholly within his head. Also, one is sometimes assured that with certain people "out of sight is out of mind." What is in their minds is therefore in sight, and cannot therefore be inside their heads.

APPENDIX A.

T. L. G., p. 105.

"She's in that state of mind," said the White Queen, "that she wants to deny something—only she doesn't know what to deny."

"A nasty, vicious temper," the White Queen remarked; and then there was an uncomfortable silence for a minute or two.

APPENDIX B.

H. S., p. 3.

"Just the place for a Snark! I have said it twice:
That alone should encourage the crew.
Just the place for a Snark! I have said it thrice:
What I tell you three times is true."

H. S., p. 50.

"'Tis the note of the Jubjub! Keep count, I entreat;
You will find I have told it you twice.
'Tis the song of the Jubjub! The proof is complete,
If only I've stated it thrice."

APPENDIX C.

A. A. W., pp. 104-105.

The Hatter had told of his quarrel with Time, and Time's refusal now to do anything he asked: "....It's always six o'clock now!"

A bright idea came into Alice's head. "Is that the reason so many tea things are put out here?" she asked.

"Yes, that's it," said the Hatter with a sigh: "it's always tea time, and we've no time to wash the things between whiles."

"Then you keep moving round, I suppose?" said Alice.

"Exactly so," said the Hatter: "As the things get used up."

"But what happens when you come to the beginning again?" Alice ventured to ask.

"Suppose we change the subject," the March Hare interrupted yawning. "I'm getting tired of this."

A. A. W., pp 145-146.

"And how many hours a day did you do lessons?" said Alice in a hurry to change the subject.

"Ten hours the first day," said the Mock Turtle, "nine the next and so on."

"What a curious plan!" exclaimed Alice.

"That's the reason they're called lessons," the Gryphon remarked: "because they lessen from day to day."

This was quite a new idea to Alice, and she thought it over a little before she made the next remark. "Then the eleventh day must have been a holiday?"

"Of course it was," said the Mock Turtle.

"And how did you manage on the twelfth?" Alice went on eagerly.

"That's enough about lessons," the Gryphon interrupted in a very decided tone. . . .

APPENDIX D.

A. A. W., p. 99.

"Two days wrong!" sighed the Hatter. "I told you butter wouldn't suit the works!" he added, looking angrily at the March Hare.

"It was the *best* butter," the March Hare meekly replied.

"Yes, but some crumbs must have got in as well," the Hatter grumbled; "you shouldn't have put it in with the breadknife."

The March Hare took the watch and looked at it gloomily: then he dipped it into his cup of tea, and looked at it again: but he could think of nothing better to say than his first remark, "It was the best butter, you know."

APPENDIX E.

A. A. W., pp. 180-187.

. . . ."Consider your verdict," he [the King] said to the jury, in a low trembling voice.

"There's more evidence to come yet, please your Majesty," said the White Rabbit, jumping up in a great hurry: "this paper has just been picked up."

"What's in it?" said the Queen.

"I haven't opened it yet," said the White Rabbit; "but it seems to be a letter written by a prisoner to somebody."

"It must have been that," said the King, "unless it was written to nobody, which isn't usual, you know."

"Who is it directed to?" said one of the jurymen.

"It isn't directed at all," said the White Rabbit, "in fact there's nothing written on the outside." He unfolded the paper as he spoke and added "it isn't a letter, after all: it's a set of verses."

"Are they in the prisoner's handwriting?" asked another of the jurymen.

"No, they're not," said the White Rabbit, "and that's the queerest thing about it." (The jury all looked puzzled.)

"He must have imitated somebody else's hand," said the King. (The jury brightened up again.)

"Please your Majesty," said the Knave, "I didn't write

it, and they can't prove that I did: there's no name signed at the end."

"If you didn't sign it, said the King, that only makes the matter worse. You *must* have meant some mischief, or else you'd have signed your name like an honest man."

There was a general clapping of hands at this: it was the first really clever thing the King had said that day.

"That proves his guilt, of course," said the Queen: "so, off with. . . ."

"It doesn't prove anything of the sort!" said Alice. "Why, you don't even know what they're about!"

"Read them," said the King.

The White Rabbit put on his spectacles. "Where shall I begin, please your Majesty?" he asked.

"Begin at the beginning," the King said very gravely, "and go on till you come to the end: then stop."

There was dead silence in the court, whilst the White Rabbit read out these verses:

"They told me you had been to her,
And mentioned me to him:
She gave me a good character,
But said I could not swim.

"He sent them word I had not gone
(We know it to be true):
If she should push the matter on,
What would become of you?

"I gave her one, they gave him two,
You gave us three or more:
They all returned from him to you,
Though they were mine before.

"If I or she should chance to be
Involved in this affair,
He trusts to you to set them free
Exactly as they were.

"My notion was that you had been
(Before she had this fit)
An obstacle that came between
Him, and ourselves, and it.

"Don't let him know she liked them best,
For this must ever be
A secret kept from all the rest,
Between yourself and me."

"That's the most important piece of evidence we've heard yet," said the King, rubbing his hands; "so now let the jury—"

"If any one of them can explain it," said Alice (she had grown so large in the last few minutes that she wasn't a bit afraid of interrupting him), "I'll give him sixpence. I don't believe there's an atom of meaning in it."

The jury all wrote down on their slates, "She doesn't believe there's an atom of meaning in it," but none of them attempted to explain the paper.

"If there's no meaning in it," said the King, "that saves a world of trouble, you know, as we needn't try to find any. And yet I don't know," he went on, spreading out the verses on his knees and looking at them with one eye; "I seem to see some meaning in them after all; *'said I could not swim—'*; you can't swim, can you?" he added, turning to the Knave.

The Knave shook his head sadly. "Do I look like it?" he said. (Which he certainly did *not*, being made entirely of cardboard.)

"All right, so far," said the King; and he went on muttering over the verses to himself:

"*'We know it to be true'*—that's the jury, of course—*'If she should push the matter on'*—that must be the Queen—*What would become of you?'* What indeed!—*'I gave him one, they gave him two!'*—why that must be what he did with the tarts, you know—"

"But it goes on, *'They all returned from him to you,'*" said Alice.

"Why, there they are!" said the King, triumphantly pointing to the tarts on the table. "Nothing can be clearer

than that. Then again—*'before she had this fit'*—you never had fits, my dear, I think?" he said to the Queen.

"Never!" said the Queen, furiously, throwing an inkstand at the Lizard as she spoke. (The unfortunate little Bill had left off writing on his slate with one finger, as he found it made no mark; but he now hastily began again, using the ink that was trickling down his face, as long as it lasted.)

"Then the words don't *fit* you," said the King, looking round the court with a smile. There was a dead silence.

"It's a pun!" the King added in an angry tone, and everybody laughed. "Let the jury consider their verdict," the King said, for about the twentieth time that day.

"No, no!" said the Queen. "Sentence first—verdict afterward."

"Stuff and nonsense!" said Alice loudly. "The idea of having the sentence first!"

"Hold your tongue!" said the Queen, turning purple...

APPENDIX F.

A. A. W., p. 56.

[Said the Pigeon to Alice:]...."No, no! You're a serpent; and there's no use denying it. I suppose you'll be telling me next that you never tasted an egg!"

"I *have* tasted eggs, certainly," said Alice, who was a very truthful child; "but little girls eat eggs quite as much as serpents do, you know."

"I don't believe it," said the Pigeon; "but if they do, why then they're a kind of serpent, that's all I can say"

This was such a new idea to Alice, that she was quite silent for a minute or two, which gave the Pigeon the opportunity of adding, "You're looking for eggs, I know *that* well enough; and what does it matter to me whether you're a little girl or a serpent?"

"It matters a good deal to me," said Alice hastily;....

APPENDIX G.

A. A. W., p. 106.

"But why [asked Alice] did they live at the bottom of a well?"

"Take some more tea," the March Hare said to Alice very earnestly.

"I've had nothing yet," Alice replied in an offended tone: "so I can't take more."

"You mean you can't take *less*," said the Hatter: "it's very easy to take *more* than nothing."

APPENDIX H.

A. A. W., p. 98.

"Then you should say what you mean," the March Hare went on.

"I do," Alice hastily replied; "at least—at least I mean what I say—that's the same thing, you know."

"Not the same thing a bit!" said the Hatter. "Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see.'"

"You might just as well say," added the March Hare, "that 'I like what I get' is the same thing as 'I get what I like'!"

"You might just as well say," added the Dormouse, which seemed to be talking in its sleep, "that 'I breathe when I sleep' is the same as 'I sleep when I breathe'!"

"It *is* the same thing with you," said the Hatter, and here the conversation dropped....

PHILIP E. B. JOURDAIN.

FLEET, ENGLAND.

THE HEBREW TITHE.

PROFESSOR DRIVER in his commentary upon Deuteronomy (p. 172) says that "the data at our disposal do not enable us to write a history of the Hebrew tithe." Conceding this to be true as regards minor details, we may yet believe it possible to indicate the main outlines of the process by which the Pentateuchal prescriptions and more recent Hebrew practice were shaped. It seems probable that we have in them a fusion of many minor ritual or ceremonial offerings and fees, originating in many ways, and known by various corresponding local names; and that the technical term "tithe" comes eventually to have a much wider application than at first, including many types of offering known originally by other names; and that in both its primitive and later use the term never had definite limitation to an exact tenth in significance.

The notion that an exact tenth was always implied underlies many earnest exhortations to greater liberality that are now current in religious literature. Serious minded persons will approve the appeals; students of history and sociology will consider them weakened by violent and needless assumptions of an inspired or divinely prescribed origin of a systematic devotion of one tenth to sacred purposes, and of the existence of such a system from the beginning of human history.

Some recent books upon this hackneyed theme collect a quantity of useful material that is uncritically dealt with.

The writers have wrought under the influence of a theory that takes for granted everything that is really to be proved. Instances of the use of the word "tenth" in connection with religious offerings are gathered from the literature of various ancient peoples, and it is immediately inferred that "the universality of the practice points to a time when the ancestors of all nations lived together and so derived the knowledge from a common source."[1] No attempt is made to examine the history of each nation, to know if the "tithe" was a feature of its whole history, or if it belongs only to a later period. It is assumed that "nearness to Eden and the Dispersion had left in their minds a deep sense of obligation to the true God."[2] Masses of citations from the Talmud, and from early Christian fathers, are all to no purpose, as neither the citations nor those who cite them examine fundamental questions. The labored efforts of rabbinical and Christian expositors to harmonize and expand prescriptions gathered from the Old Testament show only too clearly the unhistorical methods and presuppositions with which they work. Like some fervent and hasty modern writers, they assumed that all the fragmentary legislation and institutions they considered were synchronous, primitive and in force throughout all the land of Israel. There seems no thought of possibly varying customs in different epochs or in different parts of the country; nor any recognition of the possibility of the existence of various unrelated methods of maintaining religious institutions, alike among Israelites and Gentiles.

The first question that is raised by such studies is, What is the actual significance of certain idioms of speech? No matter how often we may find words from a root implying "ten," in connection with the support of religious institutions; is such technical terminology to be construed liter-

[1] Lansdell, *The Tithe in Scripture*, p. 18; cf. *The Sacred Tenth*, Vol. I.
[2] Babbs, *The Law of the Tithe*, p. 16.

ally, or is such construction contrary to the idiomatic usage of the language in which the expressions occur? We are familiar with the fact that some specific numbers in Hebrew, like seven, forty, twelve, one hundred, one thousand, may be used as general terms instead of in precise or literal signification. How far does such usage of definite numbers extend? How much arithmetical knowledge belongs to a primitive people?

Palgrave tells us[3] that the simplest computation in addition of cash, etc., involves an immense difficulty for the Beduin. A council of the wisest heads in the tribe labors with the Herculean task, and the computation may be repeated a dozen times, ere they are sure whether it is 20 or 30 piasters the sheikh has in his hand. "Even among villagers in Nedjed computation in an artificial medium surpasses the ordinary range of human faculties."[4]

Dr. Peters, in his work at Nippur, found the local Arab sheikh demanding employment for fifty of his tribesmen; that number was necessary to avoid jealousy since there were five sheikhs in the tribe, and ten men must be employed from each. He could not understand that thirty was as capable of equal division by five as fifty. His primitive faculties could count only by tens.[5] We thus face at once the difficulty of finding anything definite about any expression whatever based upon a decimal system of notation. Primitive people who have learned to count as far as their fingers will carry them, may eventually get hold of "one finger" or "tenth," as their smallest convenient fraction, so that the expression "tithe," or "tenth," may mean only fraction, portion, percentage. An illustration of such uncertainty may be cited here from the Old Testament. The champions of a divinely ordained literal tenth frequently cite Abraham as paying "tithes" of the spoils of war (Gen.

[3] *Central and Eastern Arabia*, p. 21.
[4] *Ibid.*, p. 369.
[5] *Nippur*, I, 245.

xiv. 20). But in the prescription for a percentage of spoils in Num. xxxi, the "tithe" proves to be one five hundredth part from some people, one fiftieth from others. The theory of an equal percentage from all persons and of all booty is clearly founded upon ignorance.

We may compare with the above limitations of Beduin mental arithmetic the following from central Africa. Declé found that some native tribes about Lake Tanganyika had numerals up to 7; then 8 was 7 + 1. By use of their fingers, they could recognize the number ten, but they had no word for it; as for multiples of ten, they had to begin all over again, counting out a new ten, and having no names for such multiples.[6]

Still more limited is the Bushman's vocabulary. Thompson[7] reports that he has words for 1, 2, and 3. He can count up to ten by adding twos and ones; thus four is two-two; five is two-two-one; six is two-two-two, and so on. He seems incapable of the mental feat of adding three at a time. It goes without saying that for such a stage of culture the idea of "a tenth" is impossible.

A. H. Savage Landor[8] finds that the modern Abyssinian, despite boasts of traditional descent from the all-wise Solomon, has very elementary knowledge of numbers. The customs officials at Baltehi could not sum up beyond ten, failing after repeated efforts. Nor were they better acquainted with writing materials than their Beduin kinsmen of Arabia. Of the Nilotic and Bantu tribes, he tells us that few Africans can count accurately beyond five. Among the Shiluk, six = "the hand and one"; seven is "the hand and two," and so on.[9]

The Bauda spoke to him of an event as occurring "two handfuls" (= ten seasons) ago.[10] Numerical inquiries

[6] *Three Years in Savage Africa*, 299-300.
[7] *Travels*, p. 238.
[8] *Across Widest Africa*, I, 64-65.
[9] *Op. cit.*, 1, 311.
[10] *Op. cit.*, II, 43.

would be answered by holding up one finger, two, or three as the case might be. Any fraction for such a people would be a poser.[11] The Asandeh can go farther: "six" is "give one from the other hand"; then "eleven" is "give one from the ground"; and sixteen is "give one from the other side," or foot; and twenty is "a man"; forty-three is "two men and three fingers." Thus, with names for the first five numerals, the divinely ordained decimal notation, or abacus of the fingers and toes, becomes serviceable.[12] And again we may recognize that "a finger" or one-fifth of a "hand" would be intelligible to such a people; but one-fifth of a "finger" would not be. Given a group of units, and they could set aside one out of each ten, if necessary; but they could not calculate a tenth of any unit.

Just this stage of culture must be reflected in the Talmudic sections on tithing,[13] where any one purchasing a number of figs in the market may not eat them without tithing: but that any one who has paid his penny may be allowed then to select his figs one by one without tithing. Similarly a man working among his olive trees may eat olives all day one by one, without tithing, but he must tithe if he collect a quantity of olives at a time. The like rule applies to a workman engaged in weeding onions, or to one gathering figs, whether the tree be in the courtyard or in the garden.[14] The fact that such prescriptions survive in late Talmudic literature testifies to the tenacity of ancient customs, and probably also to a general popular crudity or incapacity for arithmetical computation in later Jewish times, as well as in early Hebrew days.

The same incapacity for dealing with a fraction appears in Arab estimates of time. Thus Captain Stigand tells us that any fraction of a year is counted as a whole one amongst East African and Suahili tribes. A year and a

[11] *Op. cit.*, I, 223.
[12] *Op. cit.*, I, 395-6.
[13] Mishna, VII, Chap. II, 4-8.
[14] Mishna, VII, Chap. III, 1, 3, 7-10.

month will be reported as "two years": a month and a day is "two months." There is a general incapacity for precise fractional computations.[15]

It is clear that the two fractions ⅕ and 1/10 which occur so often in ancient prescriptions arise from counting upon the fingers of one or both hands. We may conjecture that ⅙, which we find occasionally, is originally one day in each working week, or quarter of the moon. In the Nippur exploration Dr. Peters reports that the Beduin chief who furnished workmen for the excavation claimed as his portion one-sixth of all wages gained by his tribesmen: and his sixth was always claimed upon the basis of a week's full work, whether the tribesmen had full time or not.[16] There does not seem to be any notion of oppression. The old chieftain could count one day in six, but he could not calculate the sixth part of a lesser period. This same limitation in computing powers is sometimes reported from the illiterate of our own land. A negro tenant has been known to rent land, agreeing upon one-fourth of the crop as the rent. In the fall, when no corn was brought in, the owner of the land inquired into the matter and was told, "Dere wuzn't but jes my three loads made." Had there been a fourth load of corn at gathering time, the negro would have had no difficulty in computing and delivering the proper rent.

With the Arab incapacity for fractions of time, we may compare what O'Donovan has told us of the Turcoman nomads.[17] Their notion of time is vague, beyond twelve months; they cannot tell accurately whether a thing happened 8, 12, or 20 years before or after a given great event. Some notable event is made a basis for small computations of time, just as we find illiterate people doing in America. They have a cycle of 12 years, each having an animal

[15] *The Land of Zinj*, 112.
[16] *Nippur*, II, 71.
[17] *The New Oasis*, II, 92, 97.

name, for calendar purposes, but it has no value for the masses who do not write. In a certain town, only one could tell the traveler that a certain neighboring town had been destroyed ninety-eight years before; all others reported 500 to 2000 years. Among Somali peoples Stigand was told by an aged Reshiat that their last fight with the Turkana was 140 years before, when the narrator was no longer young, but not quite so old as now.[18]

With this incapacity of primitive peoples to deal in fractions and the vague generality expressed by the term "tenth," we may compare the purely nominal character of a census and of social organization expressed in tens. The "tithingman" is a familiar figure in English and in early American colonial life, but he had nothing to do with the collecting of dues of any kind. He was merely the responsible head of a theoretical group of ten—the equivalent of the "captain of ten" of Hebrew literature. A similar figure in ancient Italy was called the decennary—or "captain of ten," and centurion or "captain of a hundred" meant a definite rank or social status quite as much as the headship of a specific company of men. The early Saxon called a town or township a hundred, and had a captain of it. Ancient Peru had the same convenient social organization. The decennary or "captain of ten" was responsible for the protection, rights, and behavior of a given group of Indian peasantry. If they failed in any respect, he suffered the corresponding penalty. The numerical method of ranking included captains of fifties, hundreds, five hundreds, thousands, tens of thousands.[19] A like social organization or nomenclature is found among Turcoman tribes. Kokand with its 60,000 people is under a *kurbashi,* or "mayor"; then come four *aksakals,* then 96 *allik-bashis* or "captains

[18] *To Abyssinia Through an Unknown Land,* p. 223.
[19] Prescott, *Conquest of Peru,* I, 42-3.

of 50."[20] In the congested city the captain of 50 is really responsible for several hundred men. In contrast we may notice the *onbashi,* or "captain of a hundred" whom Peters styles "corporal," he being next above the private in military parlance.[21] This captain of a hundred pays $14.00 to secure promotion to the next rank, which we would call "sergeant." Such conventional titular dignities may easily mislead a stranger. The relative rank of various petty sheikhs is expressed by exaggerating the numerical strength of their following. A very slight increase in self-esteem with the aid of a few shekels might advance some Hebrew "captain of ten" to be a "captain of a hundred." A comfortable room for Elisha would easily have advanced the "great woman" of Shunem to dignity at court or in the social circle of the "captain of the host." There is really nothing of actual numerical strength or systematic organization described as coming from the nomad chief Jethro, in Exodus xviii. 21-22, or suggested by Saul in 1 Sam. xxii. 7, or by Samuel in 1 Sam. viii. 12. Nothing is expressed but the relative rank of various petty sheikhs.

This use of numbers to express family or rank has many illustrations in the Old Testament. The word Aleph, "ox," or "thousand," is very common in the sense of family, division, or clan, one of the larger sections of a tribe, as in 1 Sam. xxiii. 23, Micah v. 2, Num. x. 36, Josh. xxii. 14, 21, 30; 1 Chron. xxvii. 1, xxix. 6. Gideon in Judg. vi. 15 says, "My thousand is the feeblest in Israel." Saul, making a like statement in self-depreciation, uses another common word for "family," in 1 Sam. x. 21. So princes of tribes or subdivisions are also called "heads of thousands": Num. i. 4, 16; 1 Chron. xxvii. 1; xxix, 6; compare also 2 Sam. xvii. 4; Num. xxxi. 4; Judg. xx. 10; Num. 7. 36; 1 Kings xix. 18; Ex. xx. 6; Deut. v. 10. The word regu-

[20] Hedin, *Through Asia,* I, 96, 206.
[21] *Nippur,* II, 309.

larly used for 1000 in Hebrew, Arabic, and Syriac means 10,000 in ancient Abyssinian or Ethiopic.

Ancient India shows the same simple method of expressing social rank, and in the institutes of Manu, compiled under Brahmin influence, the higher officials are described as supervising ten, fifty, or a hundred villages, just as in the familiar parable of the pounds in the New Testament faithful service is rewarded with rulership of five or ten cities.[22] Present-day conditions in Arabia are similar, and numerical exactness in such estimates is never attained. The lieutenant or deputy at Kheybar, counting the villages under his supervision, reaches ten by counting on his fingers, and promptly loses himself and gets into the thousands.[23] Assyrian bas-reliefs show us also officials coming before the king, bearing in their hands tiny models of fortified towns, of three, four or more turrets—evidently implying distinctions in rank and responsibility; but the technical nomenclature in connection with these is as yet uncertain. Similarly the prowess of a warrior is expressed in a liberal use of tens. James Morier describes a petty Khan in Persia reporting a skirmish in which his little squad has fled from a small band of Russians: the enemy were asserted to be 50,000 strong, and 10,000 or 15,000 of them were killed. The report is excused by the fact that "these letters must travel a great distance" (and so no strict inquiry will be made) and "it is beneath the dignity of the Shah to kill less than his thousands and tens of thousands."[24] This recalls the feminine strophe and antistrophe

> "Saul hath slain his thousands
> And David his tens of thousands"

which, in Oriental conventions, gave Saul good reason to

[22] *Sacred Books of the East*, XXV, 235.
[23] Doughty, *Arabia Deserta*, II, 134.
[24] *Hajji Baba*, 222.

be suspicious or indignant. It was a grave breach of decorum, to say the least, and David was held responsible for this misconduct of his admirers, and was inferred to be plotting for the throne.

Passing from honorific uses of decimal notation, we may consider the vagueness of efforts to estimate large numbers, all over the east. The "thousands" reported slain in Beduin battles are reduced by critical inquiry to two or three; and these perhaps only wounded, not killed.[25] Barth in his travels in the Sahara was told that a great salt caravan, which citizens traveling made special effort to join for their own safety, consisted of 10,000 camels. On reaching the salt works, he found the "saltpackers" to be only 200. Adding the merchants and the troop of travelers who joined the caravan for the sake of company, the whole company was less than 2000.[26] So, in Arabia, Doughty found 1000 camels to be hardly one-tenth of that number,[27] and a booty of 13,000 camels he finds to be about 130.[28] It is fair to make a like reduction in the herds attributed to Job. "Sheep without number" as an idiom only expresses the herdsman's inability to count; equally meaningless comparisons are to the stars of the heavens, or the sands of the sea. A troop of 2100 horses Doughty finds to be about 210. All estimates of village or clan population he found usually multiplied by ten: on the other hand the fighting strength of a Beduin tribe is regularly counted as one-tenth of the whole.[29] Opponents are dealt with as liberally. The Aneyza are assailed by "1000" lances and lose "200" men—subject to ninety percent discount.[30] The sheikh Zamil musters his forces for a fray; he writes "600 camels"; the forces of his allies number "300 camels (carrying two men each) and 200 led horses."

[25] Palgrave, *Op. cit.*, 23.
[26] Barth, *Travels* (Am. edition), pp. 114, 127.
[27] *Op. cit.*, II, 400.
[28] *Ibid.*, 427.
[29] Palgrave, *Op. cit.*, 299.
[30] Doughty, II, 43.

Zamil sets out next day with "more than 1000" of the town, which Doughty says might have been 200 men. The sheikh called for twice what he had expected to get and the public reported his demand to be twice as large as it was. So a Turkish expedition against the restless town of Jowf was reported by scouts to be "40,000 men, their companies without number." A member of the expedition told Doughty their actual force was 70 irregular soldiery, with a troop of armed servants.[31]

Similarly Hedin reports a curious Chinese habit of reporting each item in the equipment of a soldier as another man; his gun, his horses, his shoes, his sword, his breeches are each so many "men."[32] Stigand in East Africa asks one of his men how many camels are in an approaching herd. "300,000." Stigand replies, "Not over 3000," and begins counting; he finds not over 1500. The man then saves the face of his estimate explaining that he counted head, hump and tail as three separate animals.[33]

In connection with this vague and meaningless use of tens in daily speech may be noted another favorite idiom for "several": viz., 300 or 360. Strabo tells us that an ancient Arabic poem celebrates "360" uses for the date-palm. Palgrave is told that the hot springs in the province of Hasa number "300."[34] The Arabs claim 300 prophets in their traditional past; there were 360 images in the ancient Kaaba, or one for every day in the Moslem year; the wise Orientalist will construe this liberally, if he allow one for every day in the month. Ward when entertained at the guest house of the village of Hashm, in Lower Babylonia, was told by the simple host that the day before he had entertained 300 horsemen—exciting astonishment and skepticism in his auditor.[35] In the Old Testament we may

[31] *Op. cit.*, II, 34, 443.
[32] *Through Asia*, I, 275.
[33] *To Abyssinia Through an Unknown Land*, 101.
[34] *Op. cit.*, p. 366.
[35] Peters, *Nippur*, I, 331.

compare Samson's 300 foxes (Judg. xv. 4); Gideon's 300 men (Judg. vii. 6), as well as the earlier thousands; Solomon's 300 concubines, (1 Kings xi. 3); and the "spear of 300 shekels" (1 Sam. xxi. 16) as illustrations of the like idiom. As for military reckonings in the Old Testament we may analyze a single one: the statement that 600,000 fighting men on foot went up from Egypt in the Exodus. Western enumerators would at once pronounce the total people to be about 3,000,000; the Beduin would see 6,000,000 as only one-tenth of a tribe are counted as bearing arms. Take the smaller figure, and consider a nomad people on the march, with flocks, herds, tents, etc. Lady Blunt[36] gives some careful estimates of the numerical strength of such tribes. The Roala encampment, for illustration, she finds to number 12,000 tents, or about 50,000 persons, and to extend for ten miles in each direction. 3,000,000 nomads similarly moving would cover 40 × 150 miles—or a strip of country, forty miles wide, from the traditional passage of the Red Sea to Mount Sinai. Passing by the amusing claim that this host passed the Red Sea in a few hours in the night, we may note the story that it is thrown into a panic by "600" Egyptian horsemen or charioteers. The whole narrative parallels the Egyptian conventional portrayal on bas reliefs of Pharaohs of gigantic size opposed by pigmies.

The data cited are illuminating as to the value of numerical expressions in general throughout the Old Testament. We should next examine the current usages of the Orient with regard to the payment of fees, taxes, imposts, religious dues; and the technical terms applied to the payments. We find at the outset that travelers in the East often speak of the payment of "tithes"; but this general use of the term is often cited erroneously by those who cling to the theory of an exact tenth. No Oriental scholar

[36] *Beduin Tribes of the Euphrates*, 344, 379, 382.

would imagine that the travelers referred to expected to be so understood. "The payment of tithes" is one of the religious duties inculcated by Islam, but in none of the prescriptions of Mohammed is a tenth the requirement. The usual term in use throughout Arabia is *zika, zakat,* which Mohammed fixed at one fortieth of all that a man had in his possession for one year. Doughty found that the emir Zamil exacted one in forty of certain kinds of corn, one in twenty of others, and 7½% of dates, while houses, shops and cattle were untaxed. Rich foreign merchants paid for trading privileges $10 per annum.[37] There was no notion of taxation of merchandise, nor of *ad valorem* duties upon imports.

The following elementary distinctions are to be carefully noted. The dominant fact about regular percentages is their connection with tillable soil. The primitive notion was never that of taxing all kinds of property, and the social system that grew up under the necessity of paying dues to the local *weli, ba'al,* or patron ancestor was one that laid all regular burdens upon the peasant or fellah. A powerful and wealthy class developed, owing nothing to the divine owner of the soil on which a house stood, after the initial foundation sacrifice, while new imposts fell upon the fellah, with each new breaking of the ground. Thus the houses, stores, shops and palaces of the city dweller represented untaxed, untithed property, and the feeling of hatred of the townsman as an oppressor of the peasantry was inevitable. Theorists who have tried to find in Joseph a pioneer of the single tax on land fail to understand that the primitive Semite did not place any tax upon land, or own any land upon which to pay tax. He paid the god, *ba'al, weli* or ancestor who was the theoretical owner, a portion of the fruits he gained by tillage. For mere permission to reside he paid nothing; he might

[37] *Op. cit.,* II, 433-4

even be dependent upon the *ba'al*'s bounty, as a beggar or poor man, or he might act as custodian and eat a share of the offering to the *ba'al*.

Doughty found Boreyda claiming overlordship of the neighboring villages of Helalieh and Bukeriyeh. The tax exacted was 5%—one in twenty—upon their annual crops.[38] Cattle, houses, town property generally, went untaxed. At Kheybar the local representative of the Turkish government was compelling the villagers to furnish firewood for his soldiery. Complaint of this as an unprecedented demand evoked the threat to seize one field in eight of their tobacco, previously untaxed. Nomads had been bringing in little cheeses to sell, but his announcement that he would have one in eight as government dues at once stopped the trade. It had not been the way of the fathers to pay dues upon them.[39] The ancient system of taxing food products, while bearing heavily on all the poor, finds it difficult to make room for a new article of food; modern vegetables would be hard to tax. The Pharisee who was willing to pay upon mint and anise and cummin was going beyond what a modern Beduin would do, but it may be doubted if he would pay anything upon his own elegant town house, while foreclosing a mortgage upon some peasant widow who had borrowed a few shekels to meet the oppressive exactions upon her scanty crop.

Palgrave[40] found that the Wahaby government (the equivalent of Pharisaism or Puritanism in modern Islam) exacted from Boreyda as religious dues one-tenth of the produce of land, while a jehad, or holy war, might call for as much as one-third. On pasture cattle the tax was one-twentieth, with a special tax on meat. On money, a tax of one-fortieth was made. On merchandise an impost of four shillings was made for each camel-load, while there were no *ad valorem* duties nor taxes on real estate, shops,

[38] *Op. cit.*, II, 414. [39] *Op. cit.*, II, 132, 208. [40] *Op. cit.*, 187-188.

etc. Officials exacted personal fees or presents at every turn. Burckhardt found that the Wahaby chieftain also claimed one-fifth of all booty captured from heretics.[41]

In modern Syria and Palestine Dr. Bliss reports that the *zakat* will approximate 2½% of the total income. Ten per cent is collected upon fruits of the land. No impost is made upon less than five camels, thirty cattle, and forty sheep. There is no tax upon house, furniture, clothing or servants.[42] In Abyssinia the tax again is on produce of the land; private landowners theoretically pay ⅒ of the crop or increase to the headman of the village who is directly responsible to the emperor for the taxes. The rent upon church lands is paid directly to the clergy; priests take their stand upon stone perches in the market places on market days, and collect dues in kind from their respective parishioners, as sales are made. Such tenants of royal or church lands are to all intents private owners, so long as dues are paid, but a deed of sale requires imperial sanction.[43] But the "tenth" reported by these observers may be anything at all —exactions being so oppressive as to depopulate some districts, according to Parkyns. Moreover, alongside the exactions of corn, another older form exists, namely the maintenance of the chief or headman of a village, or of a local priest, by tilling his land for him. This system is still common in many lands, where the exactions of an outside authority have not destroyed the early village commune.[44]

In Algeria the nominal claim of the government, civil or ecclesiastical, was one-tenth. The technical term *ashur* or "tenth" was applied to the tax on grain only; the term *zakat,* familiar in Arabia and Palestine as a general term, was here limited to the tax on flocks, and was one in a hun-

[41] *Bedouins and Wahabys,* II, 157.
[42] *Religions of Modern Syria and Palestine,* 216.
[43] Skinner, *Abyssinia,* 148-9; Hayes, *The Sources of the Blue Nile,* 163.
[44] Parkyns, *Life in Abyssinia,* II, 190ff.

In the Mishna are probably survivals of the conception of dues or offerings to a local genius of the land. Thus figs offered to one in a public place are not subject to tithe, but an owner of figs, seated at home, must pay tithe on what he himself has gathered to eat. Hawkers or merchants may eat of their figs on the high road but must pay tithe if they stop with them anywhere for the night. This may be parallel to the introduction fees familiar at crossing a frontier into alien territory. The buyer of imported corn is desirous of knowing if it has paid tithe; if not, he himself pays on what he buys. Again fruits growing in a court-yard instead of a garden may be eaten without tithing, but if the tree stand in a garden, one must tithe, though he gather his supply from branches hanging over in the court-yard. Trees on the borders of the Holy Land are scrutinized in like manner. One must tithe if the trunk of the tree stands on the Holy Land, even though he pluck from branches hanging over the frontier. In Jerusalem and the cities of refuge the problem of tithing such trees is to be determined by the direction of the branches. If you drink wine, leaning over the wine press, no tithe is due, as you do not remove the wine from its place before drinking, or if you gather fruit to store or bury in the field, you do not tithe. If you bury figs in the ground to eat on the Sabbath, you cannot take them out of the ground without tithing them. If you pull radishes or turnips out of the ground to transplant, you must tithe them. Fruits placed in court yards, or watchtowers, sheds, and summer-houses, are not subject to tithe—probably because they are not in contact with *ba'al* land. In case of the "second tithe" as it is called (Deut. xiv. 22-27) the money for which it is sold may not purchase slaves, servants, lands, or unclean animals—pointing again to the exemption of these, as in modern Arabia, from any tithe. Hog raising would evidently be a profitable, duty-free occupation; modern sticklers for

a "Mosaic tenth" would do well to consider it. One may purchase with the tithe money a clean wild animal for his banquet; but he does not tithe its skin—which only means that from time immemorial it had not been customary to do so.

Thus the data from Semitic lands show that an effort to estimate offerings to a god in tenths comes with the attainment of settled agricultural life. Not earlier than mercantile and agricultural life could there be a system of weights and measures. Writing is scarcely earlier than the necessity of keeping some business accounts. Pastoral peoples to-day are still illiterate: "We are the Beduw, we do not read," would be said to Doughty when he exhibited his credentials.[50] No exact arithmetic belongs to this stage of culture in any land. No hunting and fishing people have ever thought of giving "one-tenth" of any animal to a superior or to a divinity. Such expressions are unknown to them. They give instead, choice cuts: the head, the right shoulder, a haunch, breast, kidneys and liver, the feet, tongue or fat. And these customs survive when domestic animals take the place of wild game, so that the Old Testament itself never suggests a "tenth" of an animal slain for food, but some conventional choice piece. In Eli's time there was not even this rule at Shiloh: but the seer or priest took "potluck," contenting himself with anything his forks fished up (1 Sam. ii. 13ff). The estimation by tenths comes only when daily life is dealing with articles in bulk or large quantity. And as regards grain offerings, we can hardly conceive of anything but the primitive first fruits or harvest offerings, so long as we deal with the period of communal stores, or with individual small crops not yet an article of extensive commerce.

In early China this fact is of ancient record. In the *Sacred Books of the East* (XXX, 70, 73) reference is made

[50] *Op. cit.*, II, 289.

to the tenth of the produce of the land devoted to the worship of the ancestors. We also find corners of the field left unreaped for the poor, and gleaners follow after the reapers, just as in ancient Palestine. But this only makes the sociologist suspect a period of transition from communal stores and public lands to individual tenure. And this is corroborated by the ancient Chinese literature itself. In the Liki[51] we read that in earlier days there was no taxing or tithing of grain, but the public fields of headmen or sanctuaries were cultivated by the adjacent villagers, each of whom had to give three days labor each year. In those days also there was no shifting of homes or sale of house and land. No elegant sacrificial robes were necessary for festal days, when first fruits were offered to the patron ancestors. In later times, with the development of silk culture, a tithe of the cocoons was set aside at harvest time to make the sacrificial robes of silk.

In Hindu sacred literature the same social transitions appear with corresponding changes in ritual and fees. In Manu which is very late, royal exigency is the measure of exactions. The king takes ⅙₀ of the increase of cattle and gold; one-sixth, one-eighth, one-twelfth of the crops, of skins and of earthenware. Of labor, he may exact one day in the month. In another section he takes one-twentieth of cattle and gold, one-eighth of crops, or in time of distress, one-fourth.[52] No rigid rate appears in any of the petty Indian states; as we might have expected.

Seeing the probable origin of the "tithe," its rigid connection with grain and agriculture, we may ask how the technical term *zika, zakat,* comes to be applied to it, and to other offerings in modern Arabia and Palestine. This word means "purity, purification." Prof. E. H. Palmer defines it as a sort of poor rate amounting to ¼₀ of all the property which a man has had in his possession for a full

[51] *S. B. E.*, 227-8, 271.

[52] *S. B. E.*, XXV, 236-7, 428-9.

year, but in Mohammed's time it was a contribution to the expenses of a war against infidels.[53] Dr. Bliss thinks the general term *zakat,* purification, as applied to all kinds of "alms" etc., in Palestine must refer to the subjective blessing of giving, and to the sanctification of the remainder to a proprietor after alms is deducted.[54] I am not able to accept this. Primitive rituals aim to deal with objective facts rather than with subjective states. The name suggests the survival of a fee paid for cleansing from some form of defilement. With Professor Palmer's statement of the purpose of the one in forty *zakat* in Mohammed's time, we may compare the fact that with the Mahdi it was the same per cent of all the booty captured in war. On the Algerian frontier it was once the technical term for tax upon flocks, as already cited: 1% on sheep, 1 in 30 on oxen, 1 in 40 on camels. In modern Arabia the Beduw will pay no "tax" but will grudgingly pay the religious *zika* to some powerful lord or local government, and this *zika* is usually 1% of their flocks, as formerly in Algeria.[55] Note that the *zakat* on flocks and herds in Palestine has also been shown to be usually one in 30 or 40. We may then find some reason to believe that the "purification" was originally the familiar and universal purification from war or defilement by dead bodies in time of war, practised by savage and semi-savage tribes in all parts of the world. Changes in religion make ancient rituals obsolete; the fees paid remain. We have an illustrative example in 2 Maccabees xii. 38-43. When his soldiers have touched the dead bodies of idolaters Judas does not have them individually undergo the ceremony for cleansing from defilement. Instead, he collects the proper fee from each man and sends the whole sum, two thousand drachmas, to Jerusalem to have a cleansing

[53] *S. B. E.*, VI, XXIII.
[54] *Religions of Modern Syria and Palestine*, 215-216.
[55] Doughty, *Op. cit.*, I, 455.

ritual there. His army is content to be purified by "absent treatment." This marks an advance upon the procedure described in Num. xxxi, just as the latter is later than the individual ritual prescribed in Num. xix. We may also consider the curious emphasis in Tobit upon the idea that "alms deliver from death, and cleanseth away all sin."[56] Compare the statement in Ecclesiasticus that he who gives alms offers a sacrifice (2 Eccles. xxxv. 2). It is clear that in a strange land, where the songs of Yahveh could not be sung and priestly ceremonies were seldom available, the pious Jew is contenting himself with the payment of fees for the service he cannot procure. The process recalls a tale familiar in American frontier life of three men in great peril; each urges the others to pray or sing a psalm but none of them can. "Well then, let's take up a collection!"

The payment of a fee becoming equivalent to the actual cleansing rite, it is easy to see that the presence of an idolater in the land may be tolerated if he will pay the necessary fees for the cleansing of the land from defilement; hence the war cry, "the Koran, tribute, or the sword!" The Beduw so understand the exaction of *zika*: "If we do not pay it they call us *mushrakin* (idolators)."[57] Pay the ceremonial fees, and you are assured of ceremonial benefits.

We have seen the impossibility of deriving a *zakat* or "purification" tax upon herds of cattle, from the offering of choice cuts of a slain animal to a divinity. As we have seen that the rate is the same as for booty taken in war, or for the expenses of a proposed holy war, we are warranted in concluding that it had the same origin; and from a period of early protest against offering captured cattle to Yahveh, as in the familiar story of Saul (1 Sam. xv. 15) we pass to a period when this plan is the law, as in Num. xxxi—thence to a period when all well-to-do cattlemen pay

[56] Tobit, IV, 10.

[57] Doughty, I, 455.

the god's proportion, to insure their nonmolestation by rulers or raiders in the name of the god. Compare David's pious blackmail of Nabal. All of this is unrelated to the primitive shepherd's spring sacrifice of the first-born of the season, to insure immunity from spooks and jinns of whimsical temperament, a practice which in some form is known among most pastoral peoples. Of the various elements mingled in modern Jewish and Moslem tithes, the paternity of some is fairly clear; and the two leading terms "tenth" and "purification" have clung with considerable steadiness to the grain offerings and fees with which they were at first associated.

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INTELLECTUAL EVOLUTION AND PRAGMATISM.

THIS essay was suggested by a delayed reading of *Pragmatism* by the late William James. The viewpoint is critical and psycho-analytical. The object is to point out factors of his problem to which Professor James seemed blind and to suggest some of the immediate causes of that blindness. As a result it is hoped that some contribution may be made toward the clarification of our thinking about evolution in the methods of thinking. Thus we also provide a rough scale for the classification of intellectual processes according to their evolutionary rank.

With this conception of mental evolution we can approach a better formulation of the goal toward which we are being impelled quite blindly. By becoming conscious of the evolutionary conditions and tendencies, as these involve intellectual growth, we insure a more perfect adjustment to the laws of our own character-development and accordingly we accelerate the natural growth by eliminating some impediments in the form of infantile emotional aversions.

This may also furnish a clue to a new history of philosophy. Where formerly men have written elaborate histories of the philosophic theories by which persons have explained and justified their temperamental attitudes toward the universe, the future historian of philosophy may devote himself more to a study of the genesis and growth of the temperament itself, which determines our philosophic

creed. That is to say, we are to prepare for a history of philosophy in its subjective aspect. With the statement of this program, which is pretentious as coming from an amateur philosopher, I will proceed with the task.

Professor James divides mankind roughly, and arbitrarily, into "tender-minded" and "tough-minded" groups according to predominant tendencies. Then he gives some characteristics of these groups. The "tender-minded" are intellectualistic, religious, free-willites, monistic and dogmatic, while the "tough-minded" are empiric, irreligious, deterministic, materialistic, sceptical, etc. The underlying causes for this divergence he believed to be temperamental differences. Others who recognize these differences of temperament have used the words introverted and extraverted, which I believe to be more illuminating as descriptive symbols for these characteristics.

TEMPERAMENT AS DETERMINANT.

Professor James assures us that: "Temperaments with their cravings and refusals do determine men's philosophies *and always will.*" This statement is strong and sweeping, including with the present also the infinite future, in its denial of evolutionary change in the relation of philosophy to temperament. Since our philosophies do change it would seem that the determining temperament must be undergoing corresponding changes. But why and how does our temperament change? Unfortunately James did not undertake to define temperament, nor to inform us about its determinants. If he had undertaken this he might have discovered that, instead of being the fundamental determinant of our philosophies, "temperament" is a mere symptom which reveals the degree of development which we have attained in our attitude toward, and assimilation of, experience in relation with objectives; and that "temperament" is but a collective name for reactions which

usually we do not understand, while our attitude toward relations with objectives is the real determinant of our philosophies. However James did not so conceive it.

I believe it evident from James's exclusion of relations with objectives, as a determinant of temperament, that he must have thought of temperament, if at all, as only a vague feeling-predisposition toward particular academic solutions of human problems, without duly searching for the determinants of these feeling-predispositions.

His statement that "temperaments" (in the above sense) "always will" control our philosophies, I believe to be autobiographical of James's most fundamental feeling-attitude of indifference or aversion to the check and justification of experience in relation with objectives. This conviction is confirmed by his endorsement of this quotation from Chesterton: "The question is not whether the theory of the cosmos affects matters, but whether, in the long view, anything else affects them." Here he comes perilously near to that idealist monism "that makes our universe by thinking it."

But in spite of his evident longing to remain consistent with these positions James is unable to do so. If temperaments do "and always will" determine men's philosophies, then, of course, it is absurdly futile to try to correct or otherwise interfere with the temperamental processes, except to secure a developmental change in temperament itself. Notwithstanding James's reluctance to accept the corrective of experience with objectives, these incorrigible and unavoidable relations have so far forced themselves into his consciousness that he is compelled to seek a compromise between these intruding objectives and his aversion to them, which, of course, resulted in a contradiction.

Although temperaments do "and always will" determine men's philosophies, yet James assures us, again speaking autobiographically, that "of whatever temperament a

professional philosopher is, he tries, when philosophizing, to sink the fact of his temperament." Why should any one make the least useless effort to overcome the unavoidable temperamental determination of his philosophy? Manifestly in James the explanation is his unwillingness to face his problem with objective realities, and so he was impelled to seek a compromise and was contented with mere verbalisms, which seem plausible only so long as considered disassociated from the real issues of his problem.

JAMES'S PROBLEM SUBJECTIVE.

Here we already have a view of James's internal conflict which also prompted the book *Pragmatism*. On the one side is the general primitive and infantile human tendency to ignore the limitations imposed on our impulses by our relation with objectives, whenever the realities interfere with the realization of our desires. On the other side are those experiences, dependent upon our relations with objectives, which ever force themselves upon our consciousness, and enforce the recognition of our limitations, or exact the price of pain for disobedience. It was this internal struggle, to protect the infantile impulses against the interference of the "noise of facts," that James was trying to end. He ended it by an evasion, and to justify that evasion he appropriated pragmatism as the "happy harmonizer."

When James undertook to rationalize his problem he made the very common error of ascribing his subjective conflict over relations with objectives, to a conflict with persons of different temperament. He thought of it as a contention with them over "methods," when all the time his real conflict was only subjective, as between his aversion to "facts" and the necessity of facing the "facts" themselves.

Surely Spencer and Haeckel have no quarrel with the pragmatic test of workability as a method for gauging the

relative accuracy of our conception of objectives. Indeed, their whole endeavor, as scientists, consisted in marshalling the greatest possible quantity of experiential "facts," in relation to which they applied the test of workability.

Where James would treat some infantile hypothesis as of the same value "so far forth," as an hypothesis checked by Spencer and Haeckel, I would seek an evolutionary standard of rating. This can be done only according to the number, variety and complexity of the conditions under which the pragmatic test is applied. To change the descriptive words and call these conditional truths "the truth so far forth," does not in the least help us toward judging between the relative approximations which our concepts attain as transcripts of reality. Neither does it solve any problem. It is only a begging of the question. However, to see, in their evolutionary rank, the conditions under which the test of workability is to be applied, is some help toward better and more accurate thinking.

CONDITIONS OF TEST OF WORKABILITY.

Professor James could not have mistaken the evasion of his problem for its solution, if his aversion to the check of experience in relation with objectives had not blinded his eyes to the real nature of the conflict between himself and such persons as see the issue either much less or much more clearly than did he.

For James, and all those having a similar internal conflict unaccompanied by any greater clarity of vision as to its essential nature, pragmatism came as a "happy harmonizer," not because it solved their problem but because it seemed to justify its evasion. Now they need not care about the relative accuracy of their concepts, as transcripts of reality, and need not be troubled about entertaining mutually contradictory ideas. For them now every concept

is true and every concept is false "so far forth." Evolution in relative accuracy is ignored.

At the outset, having repudiated evolution in our attitude toward experience in relation with objectives, or repudiated objectives altogether, there can be no varying degrees in the accuracy of our concepts. All controversy, conflict, contradiction concerning our acquaintance with objectives can be henceforth "so far forth" ignored. In such a system there can be "no prejudices," not even against the conclusion derived by consciously excluding a part of the pertinent evidence.

No; the conflict was not, as James conceived it, a conflict over the pragmatic method but a difference in attitude toward the "facts" of our experience in relation with objectives. In other words the essence of the controversy is not one over the pragmatic test of workability as such, but hinges on the conditions under which the test is to be applied; that is, a difference in aversion to or craving for, and in the multiplicity of such, in their application to the test of workability. Of course this evolutionary aspect of our relation with objectives, and objective conditions for the test of workability, can have no existence for those whose intraversion is so obsessing as to inhibit the recognition of any possibility of relations with or the existence of objectives.

Having now pointed out the how and the why of James's failure to see the true factors of his problem, we may proceed to an inquiry as to *what* he failed to see in consequence of his initial shortcoming. James's feeling-aversion to these experiences with objectives, which checked his temperamental predispositions, necessarily made him an inefficient observer of such experiences.

Perhaps there is need for having some statement as to what is meant by an inefficient observer. Superiority as an observer is measured by the relative minuteness and

multiplicity of the relations and aspects in which we discern an observed object. Let us apply this to James's observation of empiricists.

He says: "Never were there so many men of a decidedly empiricist proclivity in existence as there are at the present day." This should have suggested to him that the race may be undergoing an evolutionary change in its attitude toward the "facts" of experience in relation with objectives. It did not suggest this, manifestly because of his introversion, that is, an aversion to many of such "facts" and a consequent aversion to the recognition of empiricist cravings for them as a product of evolution later in the scale of development than was his own.

Notwithstanding this, James crudely saw and pointed out several degrees in empiricist evolution, only he saw those stages as dissociated phenomena and without clarity as to detail and without any evolutionary or causal elements of unification. It is because he did not give adequate attention to detail, nor see these evolutionary relations, nor any other element of unification, that I characterize him as a relatively inefficient observer. Let us now study the determinants of temperament which James overlooked.

TECHNIQUE OF MENTAL GROWTH.

The infant's pain from contact with a hot stove may produce something more than a mere effect. It may register in consciousness, and then synchronously and conjunctively there is registered some imagery of the associated stove. Let us call this association an affect-object. Perhaps later comes some understanding of the behavior which brought the unhappy result and this suggests some idea of causation.

Here the important thing to remember is that the affect-object is the registered, indissoluble entity of consciousness, which now becomes a new and independent determinant,

which modifies every subsequent result arising from new relations, especially those bearing some analogy to the hot stove.

Another illustration. Contemplate the status of a person suffering emotional disturbance because of sexual experience, or craving, with the mental associate of fear. The conflict is one between bodily craving, satisfied or unsatisfied, and a conflicting craving for social approval. Such repressed emotions produce involuntary, defensive or compensatory reaction. Hence such individuals, solely because of these existing affect-objects, react more intensely and differently than a more healthy-minded person. The former, as defenses to the self-accusation of what is called "conscience," must denounce with absurd extravagance all those ideas or acts which are associated with his own fault. The emotionally undisturbed, and so more healthy-minded person, has no similar incentive to intensity of moral judgment. The reason is that the idea, or act under observation, has a different group of associations in his existing affect-objects. Unlike the puritan, he has no unconscious or conscious associated self-accusing shamefulness to over-determine his defensive or compensatory reactions.

Some persons see a growth in the number of objects with which we have experience but fail to see the mechanism of their cohesion within the ego, and so remain unconscious of any evolutionary process in our attitude toward objectives, which attitude, derived from past experiences, might be the determinant of their attitude toward and of their valuation of further experience. In other words, they fail to see that part of the technique of our mental growth in virtue of which past experience in relation with objectives becomes the material for a conscious induction, supervising the experiential materials by which future intellectual growth is best to be achieved.

This conscious effort and guidance toward future intellectual evolution I conceive to be a late product, a growth to be achieved and desired. This neglected factor of our intellectual evolution I conceive to be a change from relatively great aversion to checks, through compromise, to a growing conscious craving for experience in relation with objectives, as a check and justification to our impulses. Finally we also experience a great craving for the most efficient method of dealing with such experiences, to the end of making our concepts always grow to a relatively closer approximation to a perfect transcript of reality.

ENERGIC MASS AS A DETERMINANT OF TEMPERAMENT.

I can believe that among new-born babes there is some difference in mere energic quantity, conditioned upon prenatal nutrition, etc. Likewise, in each individual the energic quantity available for objective relations is again conditioned upon the individual's size and the changing efficiency with which his system performs its nutritive and scavenger functions.

In infants and adults alike the energic quantity is one determinant of the aggressiveness with which they attack the immediate environment, as well as a determinant of the pleasure-pain results derived therefrom. We have already agreed to call this related existence of ego-energy and objective, producing an affect, the affect-object. This newly established consequence of related existence, this affect-object, brings new conditions into the future reactions of the individual, just as water has reactions of its own, different from those of either oxygen or hydrogen, separately, or in mere mechanical mixture. These affect-objects now are a new determinant of our relative and varying craving or aversion, as to further experience with some or with all objectives. Thus we see that mere difference of energy-mass, by being one determining element of

the character of the earlier affect-object, indirectly and remotely become an important factor in determining whether we shall belong to the introverted or extraverted type.

But this is not the only factor. Immediately after parturition the human element of the infant's environment also begins to play a part. Parental and social demands create artificial lures and restraints, which tend to inhibit or compel a particular choice, and so become a determining factor toward the enlargement of some relations with some objectives and the curtailment of others.

Thus it may occur that an infant with much energy, but reared in a "sheltered" existence, may develop great capacity for explaining and justifying the defective mental products of excessive introverted attention. The resultant and relatively imperfect concepts of reality are projected into the outer world, and become determinants of conduct, in competition with the products of more extraverted attention. Alongside of this, the environment, in relation with the affect-objects already existing, may impose a fearful attitude and aversion to the check and justifications of new experience with objectives. Such persons may retain all through life the infantile dread of shock, the infantile tender-mindedness, that is, relative introversion, and the philosophic predispositions which it determines.

On the other hand, another infant with much less initial energy-mass but with more freedom, or more artificial coercion toward entering into relations with objectives, may develop great ability for comfortable and conscious adjustment with objectives and accordingly it develops a relatively intense craving for experience with objectives. Here then, in spite of relative deficiency of energic mass, the infant develops into the tough-minded, the extraverted type. I think that Herbert Spencer was probably an example of this.

In these illustrations we see that temperament is our attitude toward experience in relation with objectives, and that both energy-mass and the number and character of the related objectives all become determinants of new affect-objects, which, by coordination with prior affect-objects, make that complex and distinctly individual *psyche*, which manifests itself in those reactions which we call character or temperament, and which differentiates us from our neighbor.

ENERGY-MASS A NEGATIVE FACTOR.

I believe that from these considerations it already appears that energic quantity is chiefly a negative determinant of the character of our affect-objects. By placing limitations upon the possible success in our conflict with objectives, it imposes a negative barrier to our future choice and development, if we endure. Within the possibility of our efficient dealings with objectives, in the first instance our choice is again limited by the content of our most immediate environment. From this unavoidable contact we derive certain results of pleasure or pain, and so are formed the first unconsciously acquired feeling-associations, or emotions; that is, we achieve affect-objects, which by some more or less crude analogy, of the mechanism and influence of which we often and long remain quite unconscious, but which, nevertheless, determine our choice among the possible new relations to objectives. Later we become more or less aware of this mechanism and proceed more consciously to make choice of our relations with new objectives, by a more critical examination of the analogue.

As this evolution progresses we see that the immediate influence of energy-mass is growing relatively less important and more clearly negative, while the remote influences of energic mass assume a positive character as choice determinant only by means of its related existence in acquired

affect-objects. So then, as we develop the more positive characteristics of temperament our determinants are relatively more dependent upon the quantity, variety and assimilation of our prior experience with objectives, that is, dependent upon the multiplicity, variety and complexity of our previously achieved affect-objects, and our consciousness of relative approaches to identity, in the analogies which influence the choosing process.

This is the important matter which escaped the attention of Professor James. It is also the most important and unfortunate characteristic of the infantile aversion to the extraverted type, and the intensity and scope of their aversion are the measure of their relative infantilism.

EVOLUTION A CHANGE IN EMPHASIS.

I believe it desirable to amplify this discussion still further. Manifestly, "temperament" has no meaning as mere static energy. It comes into significant existence only so far as resultant distinguishable activities are manifested. But this means that it has significance and classification only according to the objectives with which the energy has been associated in self-expression, that is, in acquired affect-objects. Manifestly the last of these associates is more controlled by the prior existing affect-objects and relatively less and more indirectly controlled by the mere factor of energic mass. In other words, it is the related objectives, in the affect-objects, which on the positive side determine the differentials of temperament, and evolution is a change in emphasis to the objectively contributed factors of the psychic content.

While varying quantities of energy may determine initial differences and cause a fluctuating development in our attitude toward objectives, I know of no evidence which suggests that any mere experiential relations either directly or immediately produce any change in the very na-

ture or quality of the energy itself. The only traceable consequence seems to be an increasing consciousness of the behavior of things as they are. The affect-object produced by a particular experience, of course, may have its very existence dependent negatively upon the energy-mass at the time, but positively the results will be determined by the prior affect-objects to which it becomes related.

In the absence of evidence, the natural law of persistence of energy also impels to the conclusion that differences of temperamental manifestations are not explicable on the basis of any modification wrought by experience in the essential nature or quality of the human energy. Therefore differences in the manner of energy expenditure consist only in differences of object to which it attaches itself and in relation with which it reacts. As we increase our experience with objectives, both in number and variety, the positive aggressive factor of selective activity becomes more conspicuous and is seen more clearly to be determined by prior experience with objectives, that is, through the established affect-objects. Again it appears that intellectual evolution is mainly a growth toward greater extraversion, that is, a growing emphasis on and consciousness of the objective factors of our infinitely related existence.

MECHANISMS OF EMPIRIC GROWTH FROM INFANTILISM.

We are lured or repelled, according to an associated feeling-tone derived from some prior experience, which now, because of some more or less remote resemblance between the two objective situations, has become associated with the present reality. As we grow in the consciousness of the mental mechanism through which the past experience controls the present reaction, the more critical do we become as to the relative perfection of the analogy by which we accomplish the transference of the dynamic interest from past experience to the present choice or act.

Moreover, through the growth of such conscious use of critical capacity, there develops a conscious interest toward securing the most perfect analogy as a basis for the more efficient transference of the imperative impulses which determine "temperament" and action. The relatively greater satisfaction, and lessening of disappointments, which is obtainable through this growth toward more carefully and more consciously determined conduct, develops to a craving for increasing experiences with objectives and a more painstaking observation of the behavior in relations with them. This in turn requires, and so induces, the demand for a greater quantity and diversity of experiences and for the better understanding of these, especially in their more remotely related pleasure and pain affects.

All this again means that at this stage we grow away from infantilism, toward relative intellectual maturity, according to our craving for and achievement in the quantity, variety and complexity of our observed experience with objectives; that is, according to the complexity of our affect-objects, and according to the growth of scope, understanding and desire for conscious relations with objectives. I need hardly add that this is also the means and mechanism for becoming conscious of interobjective relations.

In the course of this growth there is an accompanying evolution in our consciousness of the advantages of a larger understanding, for a better adjustment to objectives, which understanding is attainable through the insistence upon a more perfect analogy, before the past experience controls the present action. Just to the extent that we become conscious of the advantages due to such a critical capacity, we also become conscious of its dependence upon the variety of past experiences, efficiently observed. Thus comes the growing craving for greater and ever growing variety of experience with objectives, as the foundation of more crit-

ical judgments and more perfect guidance in the solution of each successive problem.

Thus in each of us the assimilated and available mental materials, that is, the affect-objects coordinated at each particular moment, will determine our attitude toward possible new experiences with objectives, and determine the result. That control becomes more and more conscious and consciously imperative as we acquire a clearer understanding of its behavior and the advantages and the sources of its power.

TOWARD THE EVOLUTIONARY CLASSIFICATION.

Our capacity for conscious advantageous adjustment to the greater variety and complexity of objective conditions, therefore, is the practical measure of our intellectual development, which is high or low according to our craving for and assimilation of experience in relation with objectives. Therefore the growth of empiric tendency, which James saw, is a later product of evolution, and the intensity, scope and consciousness of its craving are measures of the extent of that evolution.

In other words, those the farthest evolved intellectually will insist upon the most exacting conditions for the application of the pragmatic test of workability. A relative introversion is relative infantilism; relative extraversion is relative intellectual maturity. If we are unconscious of these processes and their results we call them intuition or temperament. If we know them we call it reasoning, that is, more or less consciously supervised induction and deduction.

From this presentation of the psychic mechanism involved in our growth toward the greater objective determinants of our activities it appears that the pleasure-pain motive for action never ceases to operate, nor decreases in influence. The only change which evolution brings about

is in the associated objectives, relation to which gives this pleasure or pain, and, of course, varying degrees of the consciousness of these factors and processes. Our development is remote from the primitive, just to the extent that we consciously seek and succeed in taking account of more remote objects and relations, remote both as to time and space, as the foundation of our present pleasure and pain affects.

Now we come to the formulation of that which James saw partially and crudely as mere unrelated phenomena. I will amplify these factors as I see them, and will arrange them in what to me seems to be their evolutionary order. As we proceed it must be remembered that the aim is to *describe behavior,* not to *define things.* The subtle and ever changing flow of human energy and its associates, which in their related existence constitute the determinants and characteristics of human purposes, do not lend themselves to accurate definition. Consequently human language is here a relatively inefficient tool for expressing such mobile and subtle relationships. It follows that readers will get my meaning only in so far as they ignore the precise and usual meaning of the words I use, in an effort to understand the behavior of the forces I am trying to describe. Perhaps this should have been said before.

Having now noted generally the mechanism involved in intellectual evolution we proceed to a closer observation of the process with a view to discovering at least roughly distinguishable stages in the growth to maturer mental methods.

In the main, the mental mechanism of the sick mind is like that of the healthy mind, with paucity of materials and relative inefficiency of infantile states of development, in the use of available materials for checking the energies toward a comfortable adjustment to the environment. In both, desire creates phantasies of wishes fulfilled. In the

sick mind a subjective conflict intensifies some desires, as compensation for other losses. The intensification of desire tends to preclude the coordination of the resulting phantasy with those experiences in relation with objectives which are appropriate to the checking and correction of the phantasmal content, so as to make it a relatively closer approximation to a correct transcript of the realities of the individual's situation.

In this condition of sick-mindedness the check of related realities is painful, and the impulse to avoid this pain tends to induce the avoidance of new relations to environment. This limitation of experience and of coordination promotes a relative incapacity for distinguishing between the varying degrees of accuracy, as transcripts of objectives, which may inhere in the phantasy. Accordingly the phantasy bears little resemblance to the objective realities and is workable only under the fewest and most simple conditions for applying a pragmatic test, and yet the phantasy—hallucination—is accepted as a guide to conduct with the same assurance as accompanies the thoughts of another which would withstand all the known checks applicable in a test of workability.

In the absence of a relatively thorough verification of our concepts these probably bear relatively little resemblance to the related objectives. It is this which makes them symptoms of sickness and infantilisms and renders them unsafe as guides for conduct in a relatively complex environment. However, with the relatively few and simple affect-objects to which coordination is permitted they stand the pragmatic test. Thus, the phantasy of the sick mind achieves the importance of an hallucination—is accepted as an accurate duplicate of objective reality, in its control of the conduct of the sick person. In consequence such a one comes to grief by failing to achieve an efficient adjustment to his environment.

Obviously, in such cases the process of securing relief is first to remove the motives for avoiding relation to objectives, as by inducing some consciousness of the causes of suffering and perhaps some hope of psycho-analysis as a remedy. So the individual is developed to desire submission to the influences of more related objectives, insuring a better social adjustment, by reducing the disparity between the hallucinatory phantasy and the more accurate concept which might result from the check of relatively larger conscious relation to environment.

THE INFANTILE ATTITUDES TOWARD EXPERIENCE.

The infantile attitude toward facts which curb desires, like that of the sick mind, is one of aversion and a consequent tendency to disregard, evade or deny them. For all humans, in so far as they retain the infantile attitude toward objectives, it is true that desire creates the wish-fulfilling thought; thought creates the "facts," and the dogma, with little or no support in experience, is its formal assertion and the attached affect-value impels to corresponding action. This is the psychic mechanism of the unconscious infantile mode of satisfying the inherent lust for power, the craving as if for omnipotence.

In the creation myths we see the universe produced by thinking it. Even the human creative desire and thinking mind is objectivized and becomes the divine creating intelligence. It seems to me this is well portrayed by the Evangelist John, and will be apparent to all who discard the acquired literal significance of words to get an understanding of the feelings and thoughts which prompted their use. Let me thus present the words of John with their determining motive interpolated in brackets.

"In the beginning was the word [formulated desire] and the word [desire] was with God, and the word [desire] was God [the Creator]. The same was in the be-

ginning with God." Thus we see that creative desire beyond human realization is objectivized as God, the Creator. Now we understand in what sense "The Seed is the word [desire] of God," and from the seed of desire all things flow. The word is made flesh. Thus infantile attitudes toward "facts" may be retained alongside a highly developed casuistic ability and so produce those highly ingenious arguments in support of a transcendental idealism.

Some of the Christian Fathers carried this infantile aversion to "facts" to the highest degree of enthusiasm by making it the cardinal virtue of their faith and creed. Some gloried in the faith which enabled them to defy "facts" by the formula, "I believe it because it is impossible." A mystic acquaintance of mine, who conceives himself the subject of divine illumination, says: "The essence of illumination is that it shall transcend and contradict normal experiences." Indeed, how could it exhibit its higher authority if it only confirmed normal experiences?

JAMES'S ATTITUDE TOWARD EXPERIENCE.

Evidently, in spite of a strong disposition toward this idealist monism, James found some facts from which he could not escape and so he resolved to secure his peace by establishing a "cordial relation" with such facts, and he points out that if others can accept pragmatism they likewise can make a virtue of partly escaping and partly accepting a painful necessity. But our relation must not be too cordial with "facts"; not such broad and deep familiarity as would entirely destroy the transcendental mystical monism. Hence his conflicts, compromises and contradictions. His love of facts is not strong enough to enable James eintirely to discard the infantile monism of his egocentric godhood. So he sticks to that and tries to accept some facts as well, and thus he becomes a pluralist and dualist as well as a monist.

This attitude of resignation for the acceptance of inescapable experience in relation with objectives, probably entitles James to enter the second class in our evolutionary scale. In this adolescent stage there is no clear and decisive preponderance of extraverted interest over the earlier introverted attention. For the want of this clear-visioned, conscious preponderance of interest, this is the stage of vacillation, evasion, compromise, confusion and contradiction. We try to hang on to the departing and grasp the coming ideals as well. We are all things to all men, free-will determinist, monistic pluralist, anthropomorphic pantheist, religious atheist, evolutionary absolutist and spiritualistic materialist, with pragmatism as the "happy harmonizer."

Let us have all the hostile facts from which we cannot escape and all the facts which seem to support our infantile, so-called *a priori* principles as well. This appears to be the attitude of James and his class of pragmatists.

The great mass of our "educated" humans probably belong in this early adolescent class as far as concerns the development of their attitude toward experience with objectives. In consequence, philosophically considered, they are as much fish as fowl.

EARLY MANHOOD ATTITUDE TOWARD FACTS.

And yet there is a considerable group of scientists who have grown to the early manhood stage of their development toward "facts." This class in our evolutionary attitude toward objectives and new affect-objects is represented by many of our scientific specialists, in the more exact physical sciences. Here, *at least in their special field of study* and in so far as they have the true scientific spirit and method, there is an aggressive quest for more intimate acquaintance with objective reality, a real craving for all the related experiential "facts," in order that every pre-

disposition and hypothesis may be subjected to the conscious check and justification of the widest possible range of relations with objectives, and the concepts thereby made to approach relatively nearer to an exact transcript of reality.

As the multiplicity of our affect-objects grows and our acquaintance with their interrelations and interdependences approaches a breadth as wide as the objectives themselves, we become more and more determined to ignore the infantile feeling-attitude or predispositions as relatively unsafe guides for the acquisition of conceptions of "truth," which are workable under any considerable variety and complexity of conditions.

THE PHILOSOPHIC TRANSITIONS.

Now we see a predominance of the tough-minded characteristics—the tendency toward a new kind of monism, energic or materialistic. The idealistic monism is less seriously entertained, and if here there is still a tendency to compromise, it is between dualism and materialistic or energic monism. The more departments of learning there are to which we aggressively and successfully apply the check of experiential relations with objectives, the more pronounced will become the leaning toward the determinism of materialistic forces. So, as we travel from dogmatism to empiricism we also travel from idealistic monism through pluralism to materialistic monism; from free-willism, through limited determinism (James and Leuba) and compromise, to complete determinism; from anthropomorphic religion through pantheism to atheism; from ego-centric godhood through spiritism to extraverted irreligion.

So also do we travel from intuitionalism to behavioristic research; from metaphysics to philosophy. Here I use "metaphysics" in the sense of a process of *reading*

principles into the objective; that is, we objectivize the intellectualizations of cravings and concepts acquired by unconscious processes, and objectively unchecked speculation, or seeming *a priori* principles; so too, I am speaking of philosophy in the sense of *reading principles out of* the universe through the conscious coordination of the greatest convenient number of possible affect-objects; that is to say, philosophy is now viewed as a synthesis of the sciences. Each stage in these developments is determined by the degree of our evolution in the changing attitude toward and emphasis upon relation with objectives, and the consequent kind of multiplicity and complexity of the affect-objects which we have synthetized in the formation of our character, or temperament, if you prefer. In this sense, then, it is true that temperaments (degrees of extraversion) determine our philosophy.

THE IDEALIZATION OF THE SCIENTIFIC METHOD.

Out of these classes, whose various characteristic attitudes toward facts have been briefly outlined, we can see a growth toward a theoretical fifth class of attitude toward "facts" which belongs to the future, for even an approximate realization. This grows out of a large consciousness of relationship not only of the individual to the objective but also of interobjective relations. Now we approach the condition in which the individual extends to his *every* impulse, opinion, hypothesis and concept that same aggressive desire to check and justify by the largest possible experience, just as the best sort of scientific specialist now does within his chosen specialty.

Now there may be a transfer of our greatest interest and energy from the prenatal or earliest infantile state of a wholly ego-centric attention, to an approximation toward wholly objective concentration of interest and attention, so thoroughly objective as always to include the subjective,

and the whole of the previous affect-objects, as a conscious part of the objectively considered materials. As we approach this latter attitude toward facts, the energic materialistic and the deterministic aspect of the universe tend to become more exclusive of the others. There are no more predispositions of infantile cravings for ego-centric godhood to compromise with.

Now our emphasis will be upon the interrelations of objectives as a still higher development. From this emphasis upon the understanding of relations there naturally comes corresponding emphasis upon the method of considering experiences. Now evolves the formulation, perfection and higher emphasis on the scientific method.

TOWARD EGO-CENTRIC PANTHEISM.

If we could actually enter into conscious relations with every part and aspect of the universe, and so achieve one all-inclusive synthetic affect-object; that is, if we could actually know *all* the "facts" of the universe the relative accuracy of our concepts would have reached an identity with reality and we would indeed be omniscient. Including all within our consciousness, we ourselves would be the pantheistic universe, and the distinction between subjective and objective would have disappeared. While this apparently cannot be realized, the natural forces seem to compel us to travel in that direction, and we might as well insure the best adjustment to the inevitable by consciously, and, as far as may be, consistently, holding the unattainable ideal as our goal, and as near as may be persist in striving toward it and always attempting the use of the whole scientific method, in every expenditure of energy.

In consequence we will see that our concepts approach a relatively greater accuracy, as transcripts of reality, just to the degree that our craving for multiplicity and coordination of affect-objects becomes realized and our ability

as efficient observers grows, by the aid of the whole best scientific method. Here we have the ideal of the scientific method (and its eternal open-mindedness) according to its highest development in the physical sciences, the application of which is now sought, not only to the specialty of the scientist, for in the more advanced stages of development it will approach an automatic application, as a check to every craving, aversion, impulse, and intellectual activity.

GLIMPSING THE EVOLUTION OF PHILOSOPHY.

In the evolution of this radical activity of the universe as manifested in the human focus, we appear first to become conscious of the fact of consciousness. In the intellectualization, or efforts at explaining causation, we gradually become conscious of an objective and then of two elements or aspects of the objective, force and matter. Then still later we discover these two aspects of the objective reflected, or as present within ourselves. That is, we seem to see energy in the movement and change of consciousness and we seem to see matter as the antithesis of thought, so we have come to think of brain as the carrier of thought force.

Those who suffer from repression, or inefficient expenditure of their energies, are prone to acquire an exaggerated consciousness of the force-aspect of things in comparison to the consciousness of its co-related material-aspect, or carrier. In consequence of this they tend to all those characteristics ascribed by Professor James to the tender-minded type. That is to say, they exalt the vital-force aspect of brain functioning and therefore are lured by metaphysics, *a priori* principles, deductive methods and the philosophy of idealistic monism.

Those who by their method of energy-expenditure acquire the wider knowledge and craving for relations with objectives, will develop more or less the characteristics de-

scribed by James as belonging to the tough-minded, and will tend to emphasize the materialistic and deterministic aspect of things.

The tender-minded, by projecting into the universe their exaggerated conception of the immaterial thought-aspect of ultimate reality, tend to view the universe as immaterial forces, or as a creation of mind. By their relative aversion to contact with the material aspect of objectives, they fail to correct, or outgrow and supplement their idealistic concepts.

The tough-minded ones, pursuant to their craving for experience in relation with the objective, will tend to manifest that development by emphasis upon the matter-aspect of the ultimate reality, and in explanation or justification such persons tend to irreligion, induction and materialistic monism.

TOWARD THE LARGER SYNTHESIS.

We do not know forces in themselves. We know only a little of their behavioristic manifestations under some special conditions. We know nothing of matter in itself. We know only some of its behavioristic manifestations. Again we know nothing of the existence of behavior of either matter or force as an independent thing. These facts, viewed in the light of our knowledge of the trend of mental evolution toward a more perfect synthesis and more comprehensive unity, point to the conclusion that matter and force are but different aspects of the same unitary ultimate reality.

We tend to think of this ultimate unity as force whenever our attention is most focused upon its movements. We tend to think of it as matter whenever the movements approach the limits of our sensibility or are too fine for our discernment. That is, the ultimate unity tends to be thought of as matter whenever we think of it as relatively or abso-

lutely static, and we tend to think of it as energy when it is thought of as being in motion. So we come to the conclusion that static force is matter, and matter in motion is force. Since neither is known in the absolute, or separately from the other, we come to the conclusion that these concepts are but incomplete views of the same thing, the incompleteness of the view resulting in varying emphasis upon different aspects which seem conflicting but really are parts of the one ultimate reality.

But we think of this ultimate unity at all only when its relations to the knowing mind are sufficiently acute to register as consciousness; that is when one or both of the dualistic aspects of the things known enter into relation with the knowing mind. Here again we are face to face with the same dual aspects of things. One part or aspect of this knowing mind we call "brain," "nerves" etc., that is, we think of it as matter. Another part or aspect of the knowing mind we call "thought," "spirit" or "soul," because we are thinking of it as dynamic, as change. But this last is always manifested in change of mental states, that is, in action. The material part or brain is the mind-force, thought of as static. Thinking is the brain-matter in action. So here we come again to that prospect, where the "soul" is the force-aspect of matter and brain is the static aspect of force, and both are but different aspects of an ultimate entity, in itself as yet unknowable.

Thus I am led to another dualism within a dualism, common alike to the knowing mind and the thing known; the subjective presenting the same inseparable aspects of force and matter (thought and brain) as we have discovered existent in the objective. At moments of most acute or clear states of consciousness we can see that this inseparability of different aspects obtains with equal certainty at the approaches to a relative state of unconsciousness. So

we may infer that they exist everywhere quite independent of our consciousness of them.

Now comes the suggestion that the ego (brain and thought), and the objective (matter and force) are again each but different aspects of the same universal and ultimate entity, and thus we reach a synthetic view which includes both the idealistic and the materialistic.

The seeming distinction between subjective and objective may mean only that a portion of the ultimate entity, which we arbitrarily segregate from the rest and call the ego, is viewed in its force-aspect and in relation with the rest of the ultimate reality which is viewed in its static, that is matter-aspect. Thus it may be that even the distinction between subjective and objective is largely an illusion, based upon mere differences of emphasis as between the energy-aspect of the ego part of the ultimate entity, thought of as in contrast with the relatively static aspect of another portion of that same ultimate entity, called the objective. The line of separation now becomes as arbitrary as that between the vegetable and animal kingdoms.

A particular subjective and a particular objective must now be considered as but different yet related foci in the distribution of the universal ultimate reality, perhaps distinguished according to a varying quantity, density, intensity, activity and consciousness, yet all one in kind. Then in consciousness *all* is related and the very distinction between the subjective and objective disappears. In the infinitude of affect-objects, that is when we achieve the theoretic, pantheistic omnipotence, all is one. In this synthetic view of these aspects and contrasts we get glimpses of a unity and a pan-monism which perhaps will be seen to include all the philosophies and all the sciences. With this hint I must rest for the present.

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THE JEWS OF CHINA.

IT is well known that Jews have been resident in China in considerable numbers[1] from a very remote period,[2] as Kohler[3] has shown from evidences of their peculiar rites preserved in connection with their synagogue. Brief notices of their existence have come from time to time before the European public.[4] It is even ascertained that some of them[5] attained an honorable rank in literature and several became ministers of provinces.[6] It seems to me that the most authentic evidence that has been left us of this class of honorable men existing among the Israelites in China is a note at the end of the first section of the law,[7] as found in the synagogue at Kae-fung, a transcript of which was sent to Europe by the Jesuit missionaries, containing the following statement: "Our master, our rabbi, R. Jacob, son of

[1] *Chinese Repository,* Vol. I, 1832, p. 8.

[2] *China, Its History, Art and Literature,* Vol. X, p. 146. "They were supposed to enter the middle kingdom about 200 B. C., during the Han dynasty, and afterward brought the Pentateuch with them from the Babylonian captivity and established themselves at Hanan in 72 A. D."

Abu-Zeyd Al-Hassan, an Arab who claims to have learned from his fellow countrymen who wrote in China in the twelfth century, states "that a massacre took place at Khan-fou in the year 878 and that 120,000 Mohammedans, Jews and Christians were killed." *Relations des voyages faits par les Arabes et les Persans dans l'Inde et à la Chine,* Tome I, p. 64.

[3] *Jewish Encyc.,* Vol. IV, p. 33, col. 2.

[4] Missionary notices found in Chinese Repositories. See also *Jewish Encyc.,* art. "China."

[5] *Chinese Repository,* Vol. XIII, 1844, p. 468.

[6] *Mih-chwang-mwán lüh,* Book IV, p. 2.

[7] This section, with a great many others, was brought to Shanghai by the two messengers of the London society who visited the colony of Kae-fung-foo in 1851. The whole are now deposited in charge of the Society for Propagation of Christianity among the Jews, in Lincoln's Inn Fields, London.

Abishai, the son of R. Eldad the (Saupher) scribe and (Melammed) teacher, finished this."[8] The date of this is about 1620. In a register of the Hebrew residents at Kaefung-foo, which was brought to Shanghai in July, 1851, and probably dates from some time in the seventeenth century, there are several mentioned as holding this office. On the first page we find: "Rabbi Jeremiah, the (Saupher) scribe, teacher, Sheloh, the son of Rabbi Akiba, the teacher, Sheloh."

In the records of the Kaon family we find: "Ezekiel; Samuel; Rabbi Issachar, Joseph, sons of Rabbi Mordecai the (Saupher) scribe; Joshua, Shalman, Rabbi Mordecai the (Saupher) scribe, son of Simeon." In the Lee family are cited: "Rabbi Reuben the (Saupher) scribe, son of Eliezer, Rabbi Ezekiel the (Saupher) scribe, son of Rabbi Shelephidem.

Interesting as such incidental notices are, they are far too meagre to satisfy the inquiring mind; and we cannot but regret the absence of fuller details, which would prove acceptable to ethnologists, to historians and to those who delight in tracing out the fortunes of the chosen people of God in the various lands of their expatriation. It seems to me however that further light can be thrown upon the existence of the Jews in China by examining some of the books dealing with travel, as well as Chinese literature itself.

Six hundred and thirty-six years before our era, or seventy years after the Jews had been driven from their land, King Cyrus published an edict throughout his empire, which included "All the kingdoms of the earth," calling the most eminent Jews that were in Babylonia; and he said to them:[9]

[8] Finn's *Jews in China*, p. 37. The original Hebrew of this note, with Latin and French translations, is given in the Prolegomena to Bagster's Polyglott Bible, p. 17.

[9] Josephus, Antiquities of the Jews, Book XI, Chap. I, par. 1.

"I have given leave to as many of the Jews that dwell in my country as please, to return to their own country and to build there the temple of God at Jerusalem on the same place where it had been before."

But a number, as is now known, remained in the Land[10] of the East, and doubtless many of them found their way into China.

Benjamin of Tudela visited eastern countries for the purpose of ascertaining the situation of the dispersed tribes. He however mentions only China.[11]

From some incidental remarks in Marco Polo's[12] travels we learn that the Jews were sufficiently numerous about this time to assert a political influence in China and Tartary. Speaking of the defeat in 1286 by Kubla Khan of the Tartar prince Nayan who had a vast number of Christians in his army, he continues: "When the Jews and Saracens perceived that the banner of the cross was overthrown they taunted the Christian inhabitants with it, saying: "Behold the state to which your (vaunted) banners and those who follow them are reduced." In the following chapter, speaking of the rites with which Kubla honored the Christian festivals at Kanbalu (Peking), he adds: "And he observed the same at the festivals of the Saracens, Jews and idolaters." Upon being asked his motive for this conduct, he said: "There are four great prophets who are reverenced and worshiped by different classes of mankind; the Christians regard Jesus Christ as their divinity; the Saracens, Mahomet; the Jews, Moses; and the Idolaters, Sogomonbarkan." Towards the end of the same dynasty we find another record, this time by Ibn Batuta,[13]

[10] Josephus, Antiquities of the Jews, Book XI, Chap. I, par. 3.

[11] Asher's edition of Benjamin of Tudela's *Itinerary*, Vol. I, p. 94, Hebrew text, or Vol. I, p. 143 of the English text. Asher, Vol. II, p. 189, remarks: "Our author however is the first European who mentions China.

[12] Marsden's translation, edited by Thomas Wright, London, 1854, p. 166.

[13] *Travels of Ibn Batuta*, Lee's translation, p. 217.

the Arabian envoy, as to the existence of the Jews in China about the year 1346. In an account of the city of Khansa (Hang-chow) he remarks: "In the second division are the Jews, Christians and the Turks who worship the Sun; these are numerous, their number is not known, and theirs is the most beautiful city."

We have an allusion to the Jews in China about this time from Galeotta Perera, an Italian gentleman, who was for many years a prisoner in that country. Speaking of the administration of justice there he says: "The Moores, Gentiles and Jews have their sundry othes. The Moores do sweare by their Mossasos, the Brachmans by their Fili; the rest likewise by the things they do worship." A native encyclopedia written at this time speaks of eight different systems of astronomy taught at various times in China, the last named being called the "Four Heavens," a theory introduced by the "Heen foreigners."[14]

Manasseh believed that part of the ten tribes crossed the great wall which divided China from Tartary and settled in the former country. He even stated that he believed that there were direct references in scripture to the migration of the Jews into China.

Basnage however states it can clearly be shown that neither the Tartars nor the Chinese are descendants from the ten tribes. He states however that the ten tribes did enter India and China and that the Jews were acquainted with these countries in Solomon's time.

Peritsol, an Italian Jew who lived two centuries ago, stated that the Jews were once powerful in China and India. He tells us that "They neither dwell in houses, till the ground nor drink wine." He also tells us how to get there.

Renandot, writing about this period, refers to the Jews

[14] *San-tsae-t'oo-hwúy*, astronomical section, p. 2.

of China as follows: "They thrive by various means, many of them cultivate the sciences, particularly philosophy, astronomy and physics."

Then came the various Roman and English missionary societies which at first brought us little and then a great deal of information about the Jews. From which information we can conclude that there must have once existed a great number of Jews in China and that they once had a great literature.[15]

Ever since 1851, when the London Society for Propagation of Christianity among the Jews sent its two representatives to Kae-fung-foo in Hanan to find out more about the Jews in that district, it has been a mooted question whether or not there are any references to the Jews in Chinese literature. There is no doubt that from the above foreign references we may conjecture that the Jews have been residents in China for a great length of time. Let us now see if we can possibly find Jewish records in China which bear out the above conclusions.

According to the testimony of one of the stone tablets in the synagogue of Kae-fung-foo the Jews first entered China during the Han dynasty,[16] and we are also told in letters of missionaries that "they came during the reign of Ming-te (A. D. 58-75) from Se-Yih, that is, 'the western regions.' "[17] It seems from what can be gathered that this western country can be none other than Persia, and that they came by the way of Khorasan and Samarcand.

There is no doubt that the Jews must have greatly increased in numbers, for I find in one place that in the year 845 Emperor Woo-tsung issued an edict and ordered the suppression of the Heen worshipers.[18]

[15] *Chinese Repository*, Vol. I, 1832, p. 8.

[16] *Chinese Repository*, Vol. XX, p. 454.

[17] *Lettres édifiantes et curieuses*, Tome XXIV, p. 62.

[18] I shall try to prove that the Heen and the Jews were one and the same people. Consult *Se-ké ts'ung-yù*, Book I, p. 19.

In the years 956 and 958 we find that the local authorities at Kae-fung-foo conferred honors upon the Jews.[19]

The next information we get about the Jews is from the tablets in the synagogues, which tell about an immigration of seventy families with tribute of western cloth.[20] The statement is attributed to the Emperor Heaon-tsung of the Sung dynasty, that "since they have come to our central land and reverently observe the customs of their ancestors, let them hand down their doctrines at Peen-leang (Kae-fung)." In 1163 the erection of a synogogue was begun at the expense of Yen-too-la, and was finished two years later while Lie Ching and Woo-sze-to were the religious heads."[21]

This may be looked upon as further corroboration of the statement above, that the Jews passed through a season of reverses and only through a special act of the imperial clemency were they allowed once more to build houses of worship.

The next reference that I find concerning the synagogue is in the latter half of the fourteenth century during the Ming dynasty when the synagogue was repaired.[22]

In 1421 the building was again repaired under the direct patronage of the emperor, and an imperial tablet of the Ming dynasty was placed in the hall of the synagogue and a royal commissioner was sent there to burn incense.[23] In 1445, owing to the fact that the front portion of the building was dilapidated, the synagogue was rebuilt, but a great flood which took place in 1461, due to the overflowing of the beds of the Yellow river, almost resulted in the de-

[19] *Mih-chwang-mwán lŭh*, Book IV, p. 2.

[20] Professor Hirth of Columbia University has suggested to me that this might be cotton, inasmuch as this fabric was introduced into China several centuries later.

[21] *Chinese Repository*, Vol. XX, pp. 454-457.

[22] *Chinese Repository*, Vol. XX, p. 457.

[23] It had also been customary for Gentiles to bestow gifts upon the Temple in Jerusalem. Cf. Schürer's *The Jewish People in the Time of Jesus Christ*, Vol. I, p. 304.

struction of the whole building. It was soon however rebuilt by the imperial permission and an additional portion was annexed in the latter half of the fifteenth century. We can well assume that there were various Jewish settlements in different parts of China, in view of the fact that the Jews of Kae-fung-foo were able to obtain rolls of the Law from Ning-hea and Ning-po instead of those destroyed by the flood. There were also Jewish colonies at Hang-chow and Peking.[24] In 1489 the building was entirely renovated and another stone was erected in commemoration of this event.

We find another tablet within the remains of the temple enclosure, dated 1512, on which is given a general outline of the religious views. At the close of the sixteenth century it is recorded that they were again deprived of their books by a fire and that they bought a roll of the Law from a Mohammedan at Ning-keang-chow in Shen-se[25] who had received it from a dying Jew in Canton. From this they made several copies. The synagogue was rebuilt by one named Chaou.

As to the fortunes of the Jews from that day to this one can find sufficient material in the records of the London Society for the Propagation of Christianity Among the Jews, the *Jewish Quarterly Review*[25a] and the *Jewish Encyclopaedia*.[26]

After examining the sources that the Jews have left us, let us now examine the Chinese literature and see what this

[24] *Lettres édifiantes et curieuses*, Tome XXIV, p. 62.

[25] This Mohammedan may in reality represent several, inasmuch as Father Trigault, in his account of the Christian missions to China, published in the beginning of the seventeenth century, states that during his time both Mohammedans and Jews were known under the general names of Hwuy-Kwuy. I have also found that the Jews and Mohammedans were also called Hwuy-tsze, and that the word for "temple" and "mosque" was the same, namely, Tsing-Chin sze (Temple of Purity). See *Chinese Repository*, Vol. XX, p. 154. Gozani, who visited China in 1704, calls the synagogue "Le-pae sze," a word which was similarly used to designate a mosque.

[25a] *Jewish Quarterly Review*, Vols. VIII, X, and XIII.

[26] *Jewish Encyclopaedia*, Vol. IV, p. 33.

neglected field contains for us on this subject. We find that during the Middle Ages the annals of China contain the mention of several foreign sects[27] by name, among which are Muh-hoo, Ta-tsin, King-keaon, Mo-ne, Mŏ-ne, Po-sze-king-keaon, Ho-shin, Heen-keaon, Teen-shin, Hoo-t'een-shin, Hoo-heen, Ho-heen, etc. Some of these sects received the name from their founder, as for instance Muh-hoo Mohammedans; some are designated according to their nationality, as for instance Ta-tsin, the Syrians; others are named from the object which they worship, such as the Heen-keaon or worshipers of Heaven.

Let us now briefly attempt to ascertain the meaning of Heen-Keaow or Heen religion, and perhaps we shall be able to find a few further references about the Jews in China. Peih-Huen, the editor of Chang-gan-che,[28] says, with regard to this sect: "I find these are the same as the (T'een) Heaven worshipers spoken of in the history of the northern Wei dynasty. But in ancient times there was no such character as Heen."[29] I have been told that there are no references to be found in the most ancient dictionaries for the character Heen, but that the earliest source containing a reference to it is the *Yuh-Peen*.[30] The character Heen is here designated as a foreign spirit. The *Kwang-yun*, of later date, gives, "a foreign spirit." In the "catalogue of official grades" there is one styled the Heen chief, pronunciation, "Hyen." The *Tseih-yun*, a work of the Sung dynasty, gives, "pronunciation, T'een; the same sound as Heen." The Yun-hwuy, a subsequent work, gives, "pronunciation, Hyen; the same sound as Heuen."

The *Luh-shoo-tung*, published during the fourteenth century, says: "In Kwan-chung (Shen-se) Hewen is called Heen. Foreigners designate Deity by the word Heen."

[27] Alexander Wylie, *Chinese and Japanese*, Vol. I, 1863.
[28] Printed in 1787.
[29] Book IX, p. 4.
[30] Published by Koo Yay-wang in 523.

The *Hung-woo-ching-yun*, about the end of the fourteenth century, gives: "A foreign spirit. The character is formed from the radical Teen (Heaven). In the Catalogue of the Tang officials there is one called the Heen chief."

The *Chung-yun-hwuy-peen*, which is a re-arrangement of the materials of the preceding, with modifications, published at the beginning of the seventeenth century, says: "This rhymes with Seen, Heen. The character is formed from the radical T'een (Heaven)."

The *Ching-tsze-tung*, published about the middle of the seventeenth century, gives, "Pronunciation, Hyen, same sound as Heen. The name of a foreign spirit. In the Catalogue of the Tang officials there is the Heen chief." The *Kang-he-tsze-teen*, published under the direction of the second emperor of the present dynasty, merely contains a summary of the preceding notes.

A new edition of the *Shwo-wan*, *Shwo-wan keae-tsze tung-shih*, gives the elements "she" (spiritual influence), and "Teen" (Heaven). The pronunciation is Heen.[31]

The Heen are also named in the *Chung-yen-sze-pae*, or "Tablet of the Chung-yen Monastery," by Shoo Yuen-ye in the first part of the ninth century. Among the various foreigners who arrived were the Syrians and the Heen worshipers.

From the above lexicographical details we conclude that Heen is the pronunciation given to the word Teen

[31] *Chinese and Japanese Repository*, Vol. I, 1863, p. 14, addition to note 8. Several dictionaries testify to the fact that an imperial officer was appointed to take charge of the office of burning incense. Yaou-Kwan, in respect to the above, says: "In a catalogue of the imperial officers I have seen one called Heen chief. When the followers of the Heen religion first arrived they were received as followers, according to the custom of the Guest Reception Hotel (Hung-loo she). As a result of this the members of this religion were subject to the authority of a tribunal. It is probable that these people arrived at the beginning of the T'ang dynasty. See Book I, p. 18. The Arabs also were compelled to have a tribunal over them, like the Jews—which is another point of similarity between these two peoples. *Relation des voyages faits par les Arabes et les Persans dans l'Inde et à la Chine*, translated by M. Reinaud, Paris, Tome I, p. 13.

(Heaven) in the province of Shen-se; that this same Heen was the generic word for Deity among western foreigners; that a foreign sect in China worshiped the being designated as Heen; and that an officer of the Tang dynasty was originally supervisor of these sects. We thus see, from the above remarks, that Heen is of recent formation, and I am told that it cannot be found in any book earlier than the sixth century, especially as we have the authority of Peih-Yuen, as we have noticed, that the words Heen and Teen were regarded as synonymous by a foreign sect. In opposition to the view stated above we have that of Yaow-Kwan[32] who traces this religion to the seventh century B. C. His chief authority is a statement taken from Too Yu's commentary on the Tso-chuen[33] which is an amplification of the Confucius history, *Ch'un-ts-ew* ("Spring and Autumn Annals").[34] The text of the *Ch'un-ts-ew* says: "In the nineteenth year of Duke He (B. C. 631) in summer, on the forty-sixth day of the cycle, the men of Choo took the Viscount Tsang and offered him in sacrifice." The *Tso Chuen* says, regarding this: "In summer the Duke of Sung caused W'an, the Duke of Cho, to offer up the Viscount Tsang on the tutelary altar by the river Suy with a desire to conciliate the eastern foreigners." The commentary on this, quoted by Yaou-K'wan, reads: "The Suy receives the Peen, and passing Chin to the east, leaving Leang-tseaow and Pang-ching, it flows into the Sze. Near this water a Heen spirit ruled, which it was the general practice to serve with tutelary worship; hence human beings were slain and offered in sacrifice." This is such a different and grossly exaggerated interpretation, and one which has doubtless

[32] In the *Se-ke ts'ung-yu*, a collection of notes critical and historical, written about the middle of the twelfth century, Book I, p. 18.

[33] This was written by Tso Kew-ming, a disciple of Confucius, and is a record of contemporary events necessary to throw light on the original chronicle.

[34] This is a history of Joo, the native state of Confucius, being the only complete work written by the sage.

been interpolated by some later hand, that it would hardly be of any use to consider it were it not for the fact that it was reproduced without comment in a modern native geography, *Hae-kwo-t'oo-che.*[35] We are glad, however, to note that the above reading is not supported by any authority. One scholar even denies the above statement.

In *Gae jch-ts-ung-ch-aou*[36] we find the following: Too Yu, in his commentary on the *Tso-chuen,* says, regarding the passage on the tutelary altar by the river Suy: "There was a Heen place of worship upon the Suy river. That is impossible, how could there be a Heen place of worship in China in the time of Duke Seang of Sung?"

Another interesting reference to the Heen places of worship is found in the *Mih-chwang-mwán lŭh,*[37] published about the twelfth century. We read there that "on the north side of the city wall of the eastern capital (Kaefung) there is a Heen place of worship. The Heen Spirit originally came from the western regions. They came here with a band of Mohammedans. The Heen worshipers greatly revere this spirit and pay great homage to him. The minister of the temple is called She, with postnomen She-Chwang. His office was hereditary[38] for a great many generations from the time of the Tang dynasty the Heen religion was prevalent in Peen and there religious teachers have succeeded one another for over two hundred years.[39]

We find that there were several other places of worship of the Heen religion in Chang-gan-che, but they do not tell us the date of their founding. The quotation in question

[35] Commissioner Lin, famous through the first war of China, collected this material from foreign sources. Wee Yuen, however, a bitter enemy of all foreigners although a great scholar, was the editor of this work. This masterpiece is, however, marred by the onesided view of its editor.

[36] This is a miscellaneous work written by Ye about the end of the Sung dynasty and said to contain a number of interesting historical data.

[37] Book IV, p. 2.

[38] There are doubtless descendants from the house of Aaron, proof of which will form the subject matter of another paper.

[39] Book IV, p. 2.

THE PILGRIMAGE.

O THOU, to whom my yearning soul I send
Of fuller knowledge of Thy truth in quest;
O Thou, to whom my knees in prayer I bend
To ask an understanding which gives rest
And peace unto an anxious, waiting heart;
O answer! is there purpose for my being?
Is need in this world's struggle for my part?
Some task for me there must be in this strife,
To lift me from the dust from whence I came;
Else why did Thy commandment give me life,
To struggle, suffer, yet to glorify Thy name?
My being craves assurance that there be
Some end, some lasting good to crown my strife,
I dare not ask that end or good to see,
But yearn to know that purpose guides my life.

When I recall the struggles of mankind,
The bitter wrongs, the evils that endure,
Doubts if indeed Thou art, steal o'er my mind,
For how can evil come from one all pure?
The years thus make my constant doubting grow,
As chaos and confusion they present;
In them no order seems, that I may know
By purpose, not by chance, man's life is bent.
Man comes into this realm of pain and tears,
His soul unasked if it desireth birth,
A way he seeks, midst many doubts and fears,

To fill his days with meaning on this earth;
He gains a goal, only to find at length
That what he sought is but an empty spoil;
Again he starts, renews his waning strength,
Tries even greater tasks, takes up new toil.
He finds his efforts vain, his struggles bare,
As on he journeys o'er life's thorny way,
And ofttimes in despair he breathes a prayer,
And pleads with Thee that Thou wilt end his stay—
Yet fears Thy answer, for he may not know
What mystery shall lie behind death's pall;
Thus, each of us shrinks from that unseen foe,
And ponders where his next dread blow may fall.

Death takes the ones whom most we need and love,
Who bring us joy and lessen all our ill,
And while we pray that we may meet above,
We do not know, we can not know we will.
There is so much our souls desire to do,
Yet little is the part that may be done,
Of all our dreams, we may but strive for few,
When death shall come, and leave those half begun.
Then will what little good we leave behind
Soon vanish, as in Spring the Winter's snow,
No sign of all our strivings will men find,
Nor trace of us will they who follow know.
Is't true then, life is but a shooting star,
Which burns with brilliance in a moment's flight
And then is gone, and leaves no trace nor sign,
As quickly lost to memory as to sight?

If Thou wilt teach me there a purpose be,
That every life has meaning in Thy sight,
Then willing, happy, will my spirit be,
No longer need I struggle in black night;

Though dark the way, and perils me beset,
Though grief and death assail me on all sides,
Strong shall I be once more the task to seek,
For which my spirit on this sphere abides.

In search of truth, I pilgrim far from home,
O'er mountain and through valley, day and night,
Still waiting for Thy token will I roam,
Nor rest will know until I see Thy light.
My journey brings me to a dizzy height,
I gaze into a canyon far below,
About me on all sides great mountains rise,
Upon whose lofty crests bright gleams the snow.
In shadowed depths a rushing torrent flows,
Down gulleys deep, with echoing roar, it falls;
'Mid castled crags the stately eagle flies,
And to its mate, with piercing cry, it calls.
All these, the bracing wind, the forest green,
The sky, the golden sun, which here combine
To make the glory of this wondrous scene,
Reveal to me Thy truth through works divine.
Then sets the sun, the darkness closes 'round,
The canyon's depth has disappeared from sight,
With eagle's scream no more the rocks resound,
But quiet reigns, and peace and pale moonlight;
The brilliant stars in myriads deck the sky,
And gleam as beacon lights of hope and cheer,
They blazon forth that Thou didst hear my cry
And bid my soul find peace, since Thou art near.
With thoughtful heart I lay me down to rest,
Beneath the stars whose light now fitful glows,
Once more I see the mountain's snow-clad crest,
And then in soothing sleep my eyelids close.

But, lighting slumber, comes an angel form,
A wondrous vision, radiant and bright,

Who speaks: "Thou pilgrim, who the truth wouldst seek,
Lo! I am sent to show the longed-for light.
Thou wouldst have meaning of man's life revealed,
Thou wouldst know if man's strivings count for aught,
Thou prayest that life's secret be unsealed,
God bids me answer what thy soul has sought.
With wonder thou didst view that scene by day,
With awe didst gaze upon the stars by night,
And now I ask, Did they themselves create
And merge themselves into that scene so bright?
The order which the universe makes plain,
That order which each human soul must see,
Is token sure, which may not be denied,
That somewhere a great Master holds the key.
Are not the earthly tools that man doth make,
Created by him for a purpose known?
Why then should water, earth, the heavens above,
All nature's gifts, be accidents alone?
They are the tools that He who rules the earth
Has made for lofty purpose of His own;
Placed in the hands of man at the world's birth,
They are the means through which His will is shown,
And since all things in nature purpose show,
Canst thou then think the Maker's highest art,
Man, who, supreme of all creation stands,
In God's great scheme plays yet a lesser part?
Man, who o'er all the sole dominion hath,
Who harnesses great nature by that right,
Who tunnels mighty mountains for his path,
And alters rivers' courses by his might,
To whom the gift of conscience has been lent,
Who, through that power, the good and true may choose,
The evil shun and wickedness resent,
And life exalt or, brute-like, may abuse.
Canst thou believe that man, creation's king,

The ruler of whatever is on earth,
Was placed thereon to live his few short hours,
And have no ordered purpose for his birth?

"This much in answer to thy prayer is sent,
But if thou further light wouldst have me give,
And teach thee how thine efforts should be bent,
I will direct thee how on earth to live—
To live, that when at last thy time is come
To bid farewell to all and then depart,
Thou mayest be at peace, and rest content,
And know for purpose thou didst play thy part.
Up to the best within thee, day by day,
Live every moment and through every hour,
And that the best grow greater shalt thou pray,
And strive thy soul shall blossom like a flower.
Where wrong exists and where oppression reigns,
Be thine the task that evil to allay,
And by the truth, as truth thou mayest see,
Be guided, and despair not in the fray.
If all the odds against thee may seem cast,
And if of striving thou mayst weary grow,
Still shalt thou smile, still must thy battle last,
Though thou be crushed, and hope no more dost know.
Let every spirit whom thou mayest meet,
Be strengthened and be bettered by thy soul,
That, leaving thee, he may ennobled be,
And better fitted to attain his goal.
If thus thou livest, thou shalt be assured
That what is asked of thee has been attained,
The final end is not for thee to know,
Sufficient be the truth that thou hast gained."

Sublimely then the radiant angel smiled,
And slowly faded from my eager sight,

The sound of singing birds breaks through my dream,
And I am wakened by the morning light.
I see the snowy mountains gleaming clear,
And watch the fleecy clouds drift in the sky,
I feel the sweeping wind against my cheek,
As joyous, happy, on my couch I lie.
For peace at last hath come unto my breast,
And I have gained the wisdom that I sought,
No more shall doubts and fears my soul perplex,
Content am I with what my vision brought.
My way in life now clear before me lies;
Thy glorious token strong has made my soul
To bear with courage all that life may bring,
As on I struggle toward the distant goal.

EMANUEL GEORGE FRANK.

DETROIT, MICH.

CRITICISMS AND DISCUSSIONS.

PRAGMATISM AND TRUTH.

To the Editor of The Monist:

I have just had the pleasure of reading your *Truth on Trial,* and have read it with very much interest. You make a very strong case against pragmatism, and yet, it seems to me, fail to appreciate its strong points. With apologies for attempting with the small equipment I possess to set right one who has with so much success devoted his life to the study and exposition of philosophy, I offer below a few objections to the arguments you present.

My first reference is to your section on pages 56 and 57 entitled "A Lie that Works Satisfactorily." It seems to me that you miss the pragmatist position there utterly.

Let us consider the question more carefully. Rothschild believes that if he can make the public believe that Napoleon has won the battle of Waterloo he (Rothschild) can take advantage of this belief on the part of the public (the investing public) to his own financial advantage. With this belief as a working hypothesis he proceeds to spread the report and at the same time so conduct his financial operations as to take advantage of those who believe the report. The judgment "works." Now Rothschild won, not because he believed a lie and the lie worked satisfactorily. Rothschild did not believe the lie that Napoleon had won at Waterloo. His dupes believed that Rothschild told the truth, acted on that belief, and found that it worked disastrously. Rothschild's judgment became true, his dupes' judgment became false. No better pragmatic example could have been chosen.

My second example is your definition of truth, especially that given on page 85. Does this fairly represent your definition? "Truth means that a subjective statement properly describes or represents an objective condition of things." Pragmatism says that "truth is a relation, not of our ideas to non-human realities, but of conceptual

parts of our experience to sensational parts" (William James, *The Meaning of Truth*, page 82). Now if by "an objective condition of things" you mean our perceptions, what Royce (*The World and the Individual*, page 95) calls sense-perceptions, then it would seem that there is no conflict between you and the pragmatists. But if you mean by "an objective condition of things" some objective reality outside of human experience, then of course you and they part company. But I find it difficult to understand just what connection with our thinking a reality entirely outside of human experience can have.

Let us take the case of the figure of the earth. There was a time in the experience of the race when it had not entered into the mind of man to conceive that the earth is a sphere. All his judgments that took into consideration the surface of the earth implicitly assumed that it was (or is) flat. And these judgments "worked." For all his purposes the earth is flat. Even to-day if a man builds his house on the assumption that the earth's surface is a plane and not the surface of a sphere, the judgment will work. For the purpose of supporting a house the surface is a plane. That is, it does not depart sufficiently from a plane surface to make any practical difference. Of course for the navigator, the engineer, the astronomer, that judgment will end in frustration, and is therefore not true. Then suppose that man had never discovered the sphericity of the earth. What could any so-called objective reality of this sort, entirely outside of his experience, have to do with the truth of his judgments? Truth is a relation of a part of our experience to other parts. If this lies outside our experience, our judgments can have no truth-relation whatever with it.

My third exception to your characterization of pragmatism is based on what you say on page 110, especially the following: "It no longer fits into the program of the 'new thought' movement, and pragmatism replaces it [the old ideal of truth] by a more elastic kind of truth which can change with the fashions, and makes it possible that we need no longer trouble about inconsistencies; for what is true to one need no longer be true to others, and the truth of to-day may be real now, and yet may become the error of to-morrow."

To the objection that according to pragmatism what is true to one need not be true to another, we may reply that experience is a social possession and that most things that are true to one must be

true to another. In most things our experiences are so nearly alike we may, and do, "postulate an irrelevance of differences." For the reason that we are social beings there cannot be anarchy in the realm of our truths. That there will be *some* difference in our truths the fact that we are also individuals will make inevitable. But in all those cases where social action is essential our truths will be nearly enough alike to work together.

To the objection that what is truth to-day may be error to-morrow, the answer is that if to-morrow is sufficiently distant in the future it may well be. Pragmatists, if a very humble member of the confraternity may speak for them, believe that *all* truth is in the process of change, some of it in very rapid process, some in a process so gradual as to be almost, or quite, imperceptible. Some truth is so well established that no change seems likely to occur in it within any time that can mean much to us. Maybe a figure may help us here. According to the geologist the whole surface of the earth, that is the land surface, is in process of weathering, from the lightest dust which the wind drives before it to the granite core of the mountain. But that does not mean that the earth's surface will all be changed to-morrow, or that the mountain climber of to-day will not find his mountain there next year, should he care to climb it again. So it is with truth. We cannot say that there is any part of it that will never be questioned and overthrown. But we can say that it is stable enough for us to find our way about in it, and be able to recognize the old peaks to-morrow.

I shall not apologize again. If this has not interested you, you have thrown it away long ago. If it has interested you, no apology is necessary.

Very sincerely yours,

M. JAY FLANNERY.

HAMILTON. OHIO.

EDITORIAL REPLY.

In answer to your first point I have to say that you are right: "No better pragmatic example could have been chosen" than the Rothschild case of making a lie work. A lie may be made to work, but that will never change a lie into truth, as according to pragmatic terminology it ought to. The Rothschild case proves that the pragmatic definition of truth is somehow deficient.

but it would none the less not excuse the use of the word "truth" under such circumstances.

This leads me to your third point in which you seem to identify truth with belief. Truth as I conceive it has nothing to do with the conception of truth. The latter may be and naturally is mostly a social experience, but this is exactly the fault of the pragmatists that they do not distinguish between truth itself and the subjective conception of truth. The former is an ideal, and what we call science is a method of work which realizes a gradual approximation to it. The attainment of truth in all completeness may be impossible, but our approach to it is not for that reason by any means either fantastic or illusory. Science holds an important position in the sphere of human activity and possesses features of greatest significance. The mistake of pragmatism is that it underrates the significance of science. But for all that, science will work on even where its significance is misunderstood.

Professor James never really understood the significance of science. He was an ingenious, highly interesting and personally lovable man. Whatever he discussed, or included in the field of his investigation, became interesting. His theories were rarely correct, generally inexact, but always fascinating. He never cared to work out his thoughts into a system that would be free from contradictions. His observations were scintillating with intellectual pyrotechnics. His success in his philosophical propositions was more due to his personal qualities than to the intrinsic value of his thought. He had a certain instinct to take the wrong and deck it out in such splendor that it became interesting to the masses, but errors in his hand, though they become beautiful and attractive, remained errors for all that and in his pragmatism his errors reached the danger point. In this connection he stood out in strong contrast with men who saw the only true philosophy in the philosophy of science which would demand of us, first a recognition of the significance of science; secondly, an understanding of the real meaning of science; and thirdly, its application to practical life.

This philosophy of science, which may also be called the philosophy of form, is based upon the objective character of our purely formal sciences. No one can learn to think scientifically who is not a master of the formal sciences. They are the basis of all objective knowledge, and thus they alone can give us the key to a comprehension of the world and the assurance of the reliability

I do not think that I have misunderstood Professor James, or missed his meaning; but I think that he formulated his definition of truth so as to point out the practical, not the theoretical, significance of truth—a method which is fundamentally wrong, if he has in mind to build up a theory of the world, of life, of scientific knowledge, and of truth in general. His preference for the practical is justifiable, but he has carried it to extremes where it is no longer applicable.

As to the second point, I will grant that there is no essential difference between the two definitions of truth, viz., that of Professor James, that it is "a relation, not of our ideas of non-human realities, but of conceptual parts of our experience to sensational parts," and the definition quoted from me: "Truth means that a subjective statement properly describes or represents an objective condition of things." The objective condition is always pictured first in sensational experience, and our conception of objective existence is based upon sensational experience. This is one of Kant's discoveries and need no more be discussed. But while sense-impressions are subjective and may be different in different individuals, there are elements in them which are stable and they constitute the basis of objective truth. These elements are purely formal features of experience which can be systematized in the purely formal sciences, arithmetic, geometry, logic. As soon as man begins to count and to measure, he thinks in objective terms. His sense-impressions may be faulty, he may be more or less color-blind, but if he makes a proper use of numbers and measures, his statements cease to be purely subjective and he furnishes data for building up scientific theories.

Pragmatists have failed to make a difference between the different statements of observation, and thus truth to them is and remains subjective. Its only guarantee of being of superior value to statements which may be less true is its practical usefulness. In spite of the importance which usefulness has in our appreciation of truth, I cannot help saying that the pragmatic definition of truth is extremely superficial.

Your explanation that for practical purposes it remains quite indifferent for a farmer to look upon the earth as a plane is quite obvious, and nobody will deny it. If we did not understand it as a matter of course, it might help us to explain how Professor James came to the conclusion that the earth of such people is really flat,

magic squares, it is applicable to other series of numbers, though the nature of the latter generally allows the employment of simpler methods. With the following method it will be noted that squares of orders $8p + 2$ do not involve any extra difficulties in constructing, as is quite common among other methods.

Fig. 1 is an example of the lowest order of squares constructed by this method, and the arrangement of its normal complementaries is shown in Fig. 2, the heavy lines connecting the two cells of each complementary couplet. This uniformity of complementary arrangement is identical for all orders concerned.

As an example we will explain the application of this method to a square of the sixth order; but first let us note the character of the pseudo-complementaries.

613	239	647	181
491	337	769	83
349	503	71	757
227	601	193	659

$S = 1680$

Fig. 1.

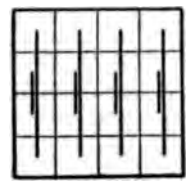

Fig. 2.

A		C		B	
8	9	24	25	40	41
7	10	23	26	39	42
6	11	22	27	38	43
5	12	21	28	37	44
4	13	20	29	36	45
3	14	19	30	35	46
2	15	18	31	34	47
1	16	17	32	33	48
A		C		B	

Fig. 3.

In Fig. 3 we have a table with n vertical columns and an indefinite number (in this case 8) of horizontal rows. This table was arranged by placing 1 in the lower left-hand cell, and numbering the cells in sequence, up in the first column, down in the second, up in the third, etc., until the table was filled. A line A—A is drawn between the first and second columns, and the line B—B centrally divides the remaining columns. The line C—C centrally divides the whole system.

It will be noticed that all horizontal pairs balanced on the line A—A have equal values. This also applies respectively to lines B—B and C—C. The pairs about the line C—C are *normal com-*

plementaries, while those about the line A—A or the line B—B are *pseudo-complementaries*. For convenience of explanation in the present writing we will classify the latter according to their rank with the normal complementaries, those about the line A—A being *minor* and those about the line B—B being *major*. It should be understood however that more than two sets of pseudo-complementaries may be used in larger squares, their number depending on the size of the square.

A complementary row of Fig. 3 is shown in Fig. 4, which plainly shows the relation of the normal and pseudo-complementaries. Each minor complementary = 17, each major = 65, and each normal = 49. Also, referring to Fig. 3, it will be noticed that the sum of one minor and two majors is equal to three normals regardless of their relative position, for example, (6+11) + (24+41) + (19+46) = (23+26) + (2+47) + (16+33).

Another feature to be observed is that each number of a pseudo-pair has its own respective normal complementary, for example, in

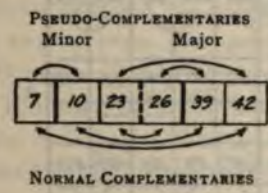

Fig. 4.

Fig. 5.

the pair 8, 9: 8 is normal to 41, and 9 is normal to 40; also, in the pair, 23, 42: 23 is normal to 26 and 42 is normal to 7.

These pseudo-pairs with their normal complementaries we will call for convenience *complementary sets*, and these are shown diagrammatically in Fig. 5. The upper line A represents a minor set, while B and C represent major sets, the minor and major classification being only in respect to the pairs shown with dots enclosed in circles, because these are used in constructing the square, while the pairs represented by dots with a line drawn through are used only when selecting sets from the prime number table.

Let us now pass on to the construction of this table and the selection of prime number sets.

Referring to Fig. 6, the horizontal rows should contain n cells (in this case 6). The number of cells in the vertical columns must be learned by experiment so as to allow for a selection of a sufficient

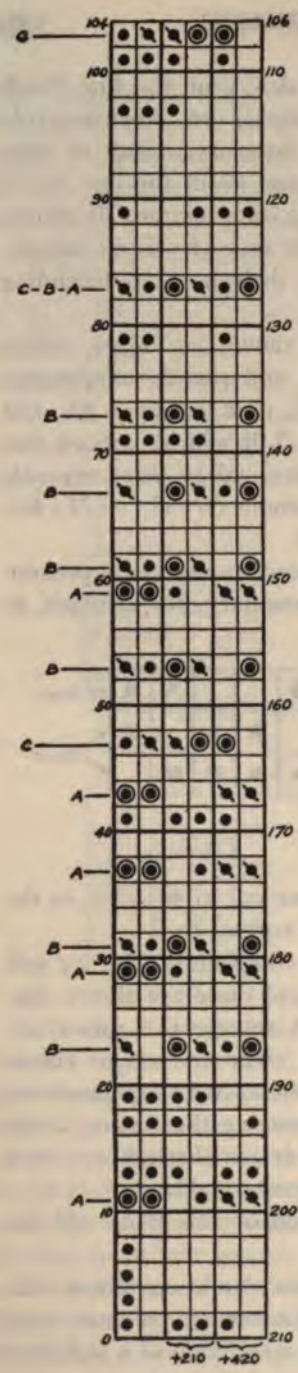

Fig. 6.

×	11	199	233	607
	29	181	241	593
	37	173	263	577
×	43	167	271	569
	59	151	277	563
			283	557 ×
			293	547 ×
			317	523 ×
			373	467

Fig. 7.

	29	181	233	607	467	373
A	263	577	173	241	37	593 A
	59	151	271	569	563	277

Fig. 8.

29	181	233	607	467	373
571	479	359	61	67	353
263	577	173	241	37	593
367	53	457	389	593	31
59	151	271	569	563	277
601	449	397	23	163	257

S=1890

Fig. 9.

number of complementary sets to build the square. The number of sets necessary is $(n/2)^2$ and of these $n/2$ must be minor sets.

There is one essential rule however governing the length of the table; that is, to create the maximum number of complementary sets the numerical advance of each double column should be a *factorial increment* or its multiple. In Fig. 6 the increment is 210, which is the product of the prime factors 2, 3, 5, and 7.

In making the table it is necessary to include the cells in which even numbers would fall, therefore this table is made for odd numbers only. To fill the table we count by odd numbers in the same procession as we did in Fig. 3, but a number is placed in its cell only when it is prime. (In Fig. 6 we have used dots instead of the numbers, their value being determined by the index numbers at the sides and bottom.)

We now select sets as indicated in Fig. 5, the letters at the left of the table indicating the respective sets. In the line C, B, A, either of these respective sets may be chosen. The dots enclosed in circles indicate the pseudo-pair to be used for construction, *and each end of these dots must have a normal complementary, which are shown with lines drawn through.*

The pseudo-complementaries are now placed in table form as shown in Fig. 7. That is, the prime numbers represented by dots enclosed in circles are transferred to a double-column table with the minor pairs in one column and major pairs in the other.

From this last table we select the diagonals. Two sets of $n/2$ (in this case three) numbers each are found which have equal sums. Two sets are indicated in Fig. 7, one set enclosed by squares and the other by circles, the sum of each being 765. Care must be taken not to have the two numbers of one pair contained in the same set, neither should there be more than $n/2$ minor pairs involved in this selection for diagonals.

One of these sets of three numbers is spaced in every second cell diagonally down to the right in the blank square, and the other set is placed diagonally down to the left as in Fig. 8.

The nine pseudo-pairs are now filled in completing the horizontal rows in which the six diagonal numbers fell, so that each row will contain one minor and two major pairs. By so doing we find that in Fig. 7 we have a surplus of pairs not needed, which are marked by an x at the side.

The arrangement of the pseudo-pairs in Fig. 8 is indicated by the curved lines which connect the numbers of each pair, the dotted

929	307	821	1327	773	1279	113	1171
1193	1453	1201	43	1217	59	1487	67
1523	271	577	1481	619	733	149	1367
1123	1489	1181	1451	1019	79	241	137
557	191	499	229	661	1601	1439	1543
157	1409	1103	199	1061	347	1531	313
487	227	479	1637	463	1621	193	1613
751	1373	859	353	907	401	1567	509

S=6720

Fig. 10.

31	599	691	3089	739	3041	757	3023	2971	809
853	1667	1979	541	2879	2791	1949	571	1367	1153
811	2969	769	3011	761	3019	479	151	1013	2767
2063	457	673	1847	2939	2731	1873	647	2053	467
1039	2741	29	2833	947	1061	2719	601	1051	2729
2111	409	3121	317	2203	2089	431	2549	2099	421
1087	2693	2477	1303	211	419	1277	2503	1097	2683
2339	181	2381	139	2399	131	2671	2999	2137	383
2297	1483	1171	2609	271	359	1201	2579	1783	1997
3119	2551	2459	61	2411	109	2393	127	179	2341

S=15750

Fig. 11.

lines indicating the minors and the full lines indicating the majors. The square is now half filled, and is completed by placing the

263	367	701	3709	643	3767	1933	2477	719	3691	3719	691
1277	1873	3727	3203	1979	1171	1949	1201	1901	1249	1667	1483
241	389	1051	3359	853	3557	863	3547	3463	347	977	3433
3673	3257	1069	2081	1913	1237	2213	937	2221	929	2027	1123
197	433	1039	3371	1091	3319	3361	1049	2039	2371	1087	3323
3671	3259	2683	467	593	2557	2341	809	2671	479	2551	599
109	521	1097	3313	3187	1223	1439	2971	1109	3301	1229	3181
3583	3347	2741	409	2689	461	419	2731	1741	1409	2693	457
107	523	2711	1699	1867	2543	1567	2843	1559	2851	1753	2657
3539	3391	2729	421	2927	223	2917	233	317	2833	2803	347
2503	1907	53	577	1801	2609	1831	2579	1879	2531	2113	2297
3517	3413	3079	71	3137	13	1847	1303	3061	89	61	3089

S=22680

Fig. 12.

167	509	673	863	331	1237
383	1259	1231	421	449	37
887	613	487	227	353	1213
373	647	773	1033	907	47
877	1	29	839	811	1223
1093	751	587	397	929	23

S=3780

601	797	239	947	43	1153
997	1249	151	431	269	683
199	599	641	1069	241	1031
1061	661	619	191	1019	229
263	11	1109	829	991	577
659	463	1021	313	1217	107

S=3780

Fig. 13.

normal complementary of each of these numbers vertically opposite to the line A—A and equidistant from it. This also completes the

diagonals, as we will now explain. The three diagonal numbers 29, 173 and 563, with their normal complementaries, would give the magic summation, but their complementaries cannot be used to complete the diagonal because of their vertical arrangement (see Fig. 2). But since the diagonal numbers $29+173+563=151+241+373$, the normal complementaries of the above two sets must be equal; therefore, $(29+173+563)$ + the normal complementaries of $(151+241+373)$ = the magic summation. By inspecting the completed square, Fig. 9, it will be found that the two diagonals are corrected as above described.

Examples of the eighth, tenth and twelfth orders are shown in Figs. 10, 11 and 12 respectively.

Fig. 13 is an example of twin squares. These are constructed by selecting a sufficient number of complementary sets to supply two squares of like summations.

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SCHENECTADY, N. Y.

THE GRAMMAR OF IDO.

The desire to establish an international language which should serve as an auxiliary means of communication between different nationalities has developed the new world language "Ido," which is practically a revision of Esperanto. A "Delegation for the Introduction of an International Language" was formed in 1901 during the World's Fair at Paris, and in 1907 an international committee of specialists chosen by them held a meeting at Paris in the College of France under Prof. Wilhelm Ostwald as chairman. After eighteen sessions the committee decided that Esperanto was the best international language but that it needed many improvements, and a new institute, the "Ido Academy," was entrusted with the task of revision. In carrying on the work they observed the following principles:

1. The alphabet is without accents, so as to avoid typographical difficulties, and remove all objection to using it in telegrams.

2. The adjectives are indeclinable as in English and the objective case is the same as the subjective, except when the object precedes the subject.

3. The derivation of words follows strictly logical rules.

4. The vocabulary must be international so as to make Ido the easiest possible speech for the greatest number of people of our modern civilization.

Since the labors of the academy have been finished, the propaganda for the international language rests with a committee, called "Uniono por la linguo internaciona," with Fr. Schneeberger, Lüsslingen bei Solothurn, as secretary, and A. Waltisbühl, of Zurich, Bahnhofstrasse 46, as treasurer.

The official organ published in Ido at Paris, and edited by the Professors Couturat and Leau under the title *Progreso*, contains all the transactions of the Ido Academy. The plan is to establish in Bern an office of the new world language which shall be competent to decide all differences and render agreement obligatory.

A German periodical edited by Pastor Fr. Schneeberger appears in monthly installments under the title *Die Weltsprache*, at a subscription price of 6 marks; a short grammar in German by Heinrich Peus, a vice-president of the Ido-Committee, has been published by the German Ido Society, called "Deutscher Weltsprache-Bund," the headquarters of which are Berlin, Charlottenburg, Waitzstrasse 24. An English key to Ido is distributed by Ward Nichols, 1306 Fitzwater St., Philadelphia; and English textbooks may also be obtained from Eugene MacPike, 135 Park Row, Chicago.

The following extract of the Ido grammar will be sufficient for our readers who wish to form an opinion of its advantages.

* * *

All letters retain the same sound throughout. There are only the five vowels, *a, e, i, o,* and *u* with the usual Italian pronunciation. There are no diphthongs. The consonants are generally the same as in English, French and German, but the following rules must be observed:

The *c* is pronounced *ts* like the German and Italian *z*, never like *s* or *k*;

s like the English *s*;

z like the English *z*;

j like the French *j* in "journal";

sh, ch, y, v, w, as in English.

The accent is always on the syllable before the last, but before other vowels *i* and *u* are pronounced as consonants, like *y* and *w*.

Thus *linguo* has the accent on *i* and *familio* on the second syllable, not on the last vowel but one.

The infinitive, ending in *ar*, has always the accent on the last syllable as in Spanish. It is derived from the Latin ending of the first conjugation, *are*, with the omission of the *e*.

The following endings mark the different parts of speech:

-o indicates the singular of a noun;

-i indicates the plural of a noun;

-a indicates an adjective;

-e indicates an adverb;

-ar (accented) indicates the infinitive of the verb.

The definite article is *la* for all three genders and both singular and plural numbers. There is no indefinite article; it is omitted as in Latin.

The genitive is formed with the preposition *di*, the dative with *a*. Thus we decline "the father":

	SINGULAR.	PLURAL.
Nom.	*la patro*	*la patri*
Gen.	*di la patro*	*di la patri*
Dat.	*a la patro*	*a la patri*
Acc.	*la patro*	*la patri*

The personal pronouns are:

SINGULAR.	PLURAL.
me, I	*ni*, we
tu, thou	*vi*, *you*
lu, he, she it.	*li*, they.

To distinguish the three genders in the third person one says:

	SINGULAR.	PLURAL.
Mas.	*ilu*	*ili*
Fem.	*elu*	*eli*
Neut.	*olu*.	*oli*.

The singular may be abbreviated to *il, el, ol.*

In courteous speech the form *vu* is used like the English "you," the French "vous" and the German "Sie."

The most important prepositions are as follows:

a, to or toward, like the Latin "ad," the Italian "a" (*ad*) and French "à";

an, near by (German "an");
ante, before in time (Latin "ante");
avan, before in space (French "avant");
apud, by the side of (Latin "apud");
che, in the house of (French "chez");
da, by, through (Italian "da");
de, away from (Latin "de");
di, of (possessive, Italian "di");
dop, behind (Italian "dopo");
dum, during (Latin "dum");
ek, out of (Greek "ek," Latin "ex");
en, in (French "en");
inter, between (Latin "inter");
kun, with (Latin "cum");
per, through (Latin and Italian "per");
por, for (French "pour");
pos, after (Latin "post");
sen, without (French "sans", Italian "senza");
sub, under (Latin "sub");
sur, upon (French "sur");
til, until (English "until");
tra, through (Latin "trans").

The infinitive of verbs ends in *r* and the three tenses are distinguished by *a* for the present, *i* for the past, and *o* for the future. Thus we have the three forms

amar, to love,
amir, to have loved,
amor, about to love.

The indicative ends in *-s* thus:

me esas, I am	*me havas,* I have
me esis, I was	*me havis,* I had
me esos, I shall be.	*me havos,* I shall have.

The conditional is indicated by the ending *-us,* thus:
me volas, I wish
me volus, I would wish
me povus, I would be able
me esus, I would be
me havus, I would have, etc.
The imperative is indicated by the ending *-ez,* thus: *Irez,* go!

parolez, speak! *audez,* hear! *facez,* do! *kurez,* run! *hastez,* hasten! *videz,* see! *mangez,* eat! *drinkez,* drink! *esez tranquila,* be quiet! *fidez,* have faith! *esez sincera,* be sincere!

The participle (adjective derived from the verb) exists in two forms, active ending in *-nta* and passive ending in *-ta*; thus: *am-anta,* loving; *am-ata,* beloved.

In combination with the auxiliary verb *esar,* to be, we can form not only active progressive tenses, but also the passive: *Me esas batanta,* I am beating; *me esas batata,* I am beaten.

Further combinations may be made as follows: *ni esis batita,* we have been beaten; ; *vi esos batita,* you shall have been beaten.

There is, however, a simpler way of forming the passive by contraction. Omitting *-ata* we may say *me am-esas,* I am loved, instead of *me amata esas.*

In the same way we may form the active progressive tenses in abbreviations by contracting the participle in *-inta* with the auxiliary verb *esar* by means of the connecting syllable *-ab.* We can contract

me esis vidinta, I was seeing, into *me vidabis*;

me esos vidinta, I shall be seeing, into *me vidabos.*

The personal pronouns are not changed in form but are declined like the nouns with the preposition *di* and *a,* thus:

Nom. *me,* I; Gen. *di me,* of me; Dat., *a me,* to me; Acc., *me,* me.

In case of inversions the accusative is indicated by the ending *n* thus:

SINGULAR.	PLURAL.
men, me	*nin,* us
tun, thee	*vin,* you
lun, him.	*lin,* them.

The impersonal pronoun "one," in the sense of the French "on" and the German "man," is in Ido *on* or *onu,* with the accusative *onun,* in case of inversion.

The possessive adjective is formed by adding the adjective ending *-a* to the personal pronoun, thus:

Mea, my; *tua,* thy; *lua,* his.

The demonstrative pronouns are as follows:

Ica, this; *ita,* that; or simply *ca* and *ta.*

The plurals *ici,* these, and *iti,* those, are also abbreviated to *ci* and *ti.*

The demonstrative pronouns of the neuter gender are *ico* or *co*, this thing; *ito* or *to*, that thing.

Relative and interrogatory pronouns are *qua*, who (masculine); *qui*, who (feminine); *quo*, what (neuter). Their accusatives are *quam*, *quin*, *quon*, whom and what.

Interrogative and relative adverbs are *kande*, when; *ube*, where; *quale*, how.

The following indefinite pronouns explain themselves:

singlu homo, each man
omna homo, every man (all)
ula hundo, some dog
irga hundo, any dog
nula amiko, no friend
kelka amiki, some friends
multa domi many houses
plura domi, several houses
tala vetero, such weather
quala vetero, what weather
tanta pekunio, so much money
quanta pekunio, how much money
sama sumo, the same amount
altra kozo, another thing
cetera homi, other people
me ipsa, myself.

The numerals are *un*, one; *du*, two; *tri*, three; *quar*, four; *kin*, five; *sis*, six; *sep*, seven; *ok*, eight; *non*, nine; *dek*, ten.

The tens are formed by the multiplicative suffix *a* combined with *dek*, thus: *dua dek*, twenty; *tria dek*, thirty; *quara dek*, forty, etc. In the same way the hundreds and thousands are formed, thus: *tria cent*, 300, and *kina mil*, 5000.

The ordinals are formed by the ending *-esma*: *un-esma*, the first; *du-esma*, the second; etc.

Fractions are formed by the ending *-ima*, plural *-imi;* thus, *duima pano*, half a loaf; *tri quarimi*, three quarters; *kin sepimi*, five sevenths.

Multiplicative numbers such as double, treble, and quadruple are formed by the ending *-opla*, thus, *duopla sumo*, a double sum; *la kinopla nombro*, the fivefold number.

Here is a list of conjunctions:

e or *ed*, and
o or *od*, or
ma, but
nam, for, because
ke, that
kande, when
se, if
pro ke, for the reason that, because
por ke, so that
sen ke, without that
per ke, through the fact that
quan kam, although.

The comparative and superlative of adjectives are formed by prefixing *plu* and *maxim* to the positive: *bona*, good; *plu bona*, better; *maxim bona*, best. The opposite forms, "less and least," are expressed by *min* and *minim*: *bela*, beautiful; *min bela*, less beautiful; *minim bela*, least beautiful.

The word *mem* (French "même") corresponds to the English "still," thus: *mem plu granda*, still greater.

The comparison of equals, "as — as" (Latin "tam — quam") is expressed in Ido by *tam — kam*, as in the following clause, *il esas tam afabla kam elu.* "He is as amiable as she."

"So — that" is expressed by *tante — ke.* "He is so tall that he is larger than all," reads in Ido, *il esas tante longa ke il super-esas omni.*

The corresponding adverbs are formed from adjectives by changing the ending *a* to *e*.

The forms of affirmation and negation are like the English, *yes* and *no*. The negative adverb "not" is *ne* and is always placed before the verb. Thus, "I do not believe" reads in Ido, *me ne kredas.*

The sequence of words in the sentence follows the logical order, as in English: *me vidis la amiko di mea patro*, "I have seen the friend of my father."

New words are formed from the roots by the aid of fifteen prefixes and forty-five suffixes.

arki- denotes higher degree (Eng. "arch-"): *arki-episcopi*, archbishop.

bo- denotes relationship by marriage (Fr. "beau"): *bo-patrino*, mother-in-law.

des- denotes the opposite (Eng. "dis-"): *des-unionar*, dissolve.

dis- denotes separation (Eng. "dis-"): *dis-sendar*, to send away.

ex- denotes former (Eng. "ex-"): *ex-urbestro*, ex-mayor.

ge- denotes taken together (Ger. "ge-" in "Geschwister"): *ge-spozi*, husband and wife (*spozo* = spouse); *ge-patri*, parents.

mi- denotes half (Fr. "mi-"): *mi-lauta*, half-aloud; *mi-apertar*, to open half-way.

mis- denotes wrong (Eng. "mis-"): *mis-kalkular*, miscalculate.

ne- denotes negation: *ne-habila*, unhandy; *ne-bela*, not beautiful.

par- denotes completion of an act (Ger. "ver-"): *par-venar*, to attain; *par-lektar*, to read through.

para- denotes protection (Fr. "para-"): *para-pluva,* umbrella; *para-fulmino,* lightning-rod.
pre- denotes before in rank and time (Eng. "pre-"): *pre-dicar,* predict.
retro- denotes back (Lat. "retro-"): *retro-sendar,* to send back.
ri- denotes again (Lat. "re-") *ri-venar,* to come again.
sen- denotes without (Fr. "sans-"): *sen-viva,* lifeless.

SUFFIXES..

-ach- denotes disparagement: *hundacho,* cur (*hundo* = dog).
-ad- denotes duration or repetition of an act: *la dansado,* dancing (*danso* = dance).
-ag- denotes action with: *martelagar,* to hammer (*martelo* = hammer).
-aj- denotes thing consisting of or made from: *novajo,* novelty (*nova* = new); *drinkajo,* beverage (*drinkar* = to drink).
-al- denotes relating to (Eng. -al"): *nationala,* national.
-an- denotes member: *skolano,* scholar (*skolo* = school).
-ar- denotes collection: *homaro,* humanity (*homo* = man).
-ari- denotes the object of an act: *sendario,* recipient.
-atr- denotes like: *sponjatra,* spongy.
-e- denotes color of: *orea,* golden.
-ebl- denotes possibility: *videbla,* visible (*vidar* = to see).
-ed- denotes quantity determined by: *glasedo,* glassful.
-eg- denotes increase; *grandega,* gigantic (*granda* = large).
-em- denotes inclined to: *babilema,* talkative.
-end- denotes necessity: *solvenda,* to be solved.
-er- denotes habitual action: *fumero,* smoker.
-eri- denotes institution: *bakerio,* bakery; *redakterio,* editorial office.
-es- denotes state or condition: *sanesar,* to be well.
-esk- denotes to begin: *dormeskar,* to fall asleep.
-estr- denotes chief: *urbestro,* mayor.
-et- denotes diminution: *dometro,* cottage (*domo* = house).
-ey- denotes place for: *laboreyo,* workshop (*laborar* = to work); *manjeyo,* dining room (*manjar* = to eat).
-i- denotes domain: *rejio,* kingdom (*rejo* = king).
-id- denotes offspring: *Napoleonido,* descendant of Napoleon; *Israelidi,* Israelites.

of gases and liquids, the concept of molecule is most fruitful and of its reality we have convincing proofs. E. H. Starling writes on "The Animal Machine and its Automatic Regulation," in which again is emphasized that, weight for weight, no man-made machine can be compared in efficiency with the numberless mechanical engines met with throughout the animal kingdom. There are two articles on aspects of nationality: A. Meillet writes on languages and nationalities; and Roberto Michels writes on the occasional lack of relationship between real and acquired nationality. Camillo Acqua gives a general review of our knowledge of the respiratory process in plants; and there are book reviews and French translations of the articles written in Italian and English.

In *Scientia* for October, 1915, Aldo Mieli maintains that, in spite of the very different views as to the nature of Greek science which are held by men of different branches of science, there is a unity in the scientific character of the ancient world, in virtue of which the development of the various sciences is due, not to the mentality of those who cultivated them, but to the special characteristics of the subject treated. P. Puiseux writes on the future of the planets, and points out that the tendency of the sun is not to contract and die out but to dilate and dissolve away. The concluding sentence is worth quoting: "The intelligence which exerts itself to view possible catastrophes with calmness is a better prop for a moral life than passive enjoyment of the present course of things." C. Lloyd Morgan has a very able and systematic discussion of the various views of "Mind and Body in their Relations to Each Other and to External Things." The only article concerned with the present problems of nations is one by Ramsay Muir of Manchester on "The Antipathy Between Germany and England." There are also reviews of books and other periodicals, and French translations of the articles in Italian and English.

In *Scientia* for November, 1915, A. S. Eddington writes on the stellar universe as a dynamical system. In the solar system dynamics has of course been successfully applied for a long time, but for the outside stellar system the study of dynamics has found but little application as yet. Charles Fabry gives the first part of an article on luminous atoms and their motions. Mario Vallauri writes on Indian medicine: medical doctrines were the consequence, in India as elsewhere, of religious theories, and the history of medicine among the Hindus is here traced from the Vedic age downward.

The only article touching on the national questions of the present is one by P. Bonfante of the University of Pavia on a possible future European confederation. Besides these articles there are reviews of books and periodicals, a chronicle, and French translations of the articles written in English and Italian.

* * *

In *Science Progress* for October, 1915, Dr. C. G. Knott gives an interesting account of the proceedings at the Napier Tercentenary held at Edinburgh in 1914 to celebrate the three-hundredth anniversary of the first publication of logarithms, and suggests that a fitting outcome of the Tercentenary would be a photographic reproduction of the manuscript volumes containing calculations of logarithms to fifteen places by the late Edward Sang. Walter Stott gives a most convincing appeal for more systematic work in the making accessible of the results of mathematical science by the thorough study of history. This appeal shows us that the value of a great part of the work of the Open Court Publishing Company will be appreciated by men of science. Some passages of Stott's article are worth quoting. "One hundred years after the publication of Laplace's *Essai philosophique sur les probabilités,* it has been translated into English by the Americans, and Lagrange's great work, the *Mécanique analytique,* which Hamilton called a scientific poem, has never been printed in English."...."Who would think of looking for valuable mathematical problems and theorems in papers bearing such titles as *The Ladies' Diary, The Gentleman's Diary, The British Diary, The Leeds Correspondent, The Northumbrian Mirror, The Liverpool Student, The Miscellanea Curiosa,* and numerous others with titles giving no index to the contents? And yet, if the history of mathematics is to be written, it is among such papers that we must search. What mathematician would think of looking into a work bearing the strange device *Instruction Given in the Drawing School Established by the Dublin Society*? And yet this book, written by Joseph Fenn (a name almost forgotten in mathematics), gives the first example of the use in the British Isles of the notation of Leibniz for the differential and integral calculus," and a discussion of the complex variable. "One of our most pressing needs is a set of works dealing with the history of special branches. Complete histories of the complex variable, of the theory of groups, of the solution of algebraic equations, of finite difference equations,

Dr. Seidenadel had not the Igorots flocking around him from early sunrise till nightfall, as we have; their talk, their songs, the sound of their gongs, constantly dinning in his ears. He had to go to them at times when they could get an interval from their show-business in American pleasure resorts, and patiently gather his material from the men and boys who became his informants, teachers and helpers.

And yet while in this there was a loss, there was also a gain. The loss was that which can only be gotten in informal, habitual contact, the things which one "picks up," as a child acquires a language, not reasoning much why or how a thing is so. The pencil and note-book plan of approaching linguistic and ethnological problems has certain drawbacks. The observer thinks he has a fact, notes it down, and proceeds to ask whether it is so. He probably elicits an affirmative answer; but the chances are at least even that the thing is not so at all, or only partly so, and under certain conditions. This is emphatically true of the Bontok Igorots, and I fancy it may be of most "primitive" people. Reliance on this method has marred the value of otherwise admirable books. It is safe to say that the ultimate facts, linguistic or other, regarding the Igorot, can never be gotten at except by one who, by actual residence among the people, gleans his information informally, but has his note-book in steady use behind a screen. One can only wish that the author, with his evident love for the work, could do this.

As it is, he has made mistakes, due doubtless to this (*quasi*) exotic though necessary character of his method. Our best language-helper (we know well most of Dr. Seidenadel's helpers and have had several of them in our employ) sits down to *The First Grammar*, and, reading a rule and examples, says occasionally, "Oh that is not so, that is not right!"

For example, we take up the volume and turn, quite at random, to p. 110 and remark that *Tekuafek* should be *Tekuafak*, the *-ek* form not being found in this verb. On p. 113 *Alitauko* should be *Alitauk*. On p. 150 *Nan soklong ay maisabfud* means not the hat which "is suspended," but which "is to be suspended." A little lower down on the same page one can say emphatically that the Bontok people *never* say *mangayak*, but *mangayag*. This is only one of many cases, however, where Dr. Seidenadel's reproductions of sounds are simply "impossible." Under "Interrogative Sentences," p. 160, my helper strenuously objects to *Aykö tinmoli siya ay*? for "Has he returned?" "Barlig or Lias men might say it that way, but Bontok never." On p. 167 *Ngagka man ken Bugti?* is declared to be meaningless, *enisibanyu ken* being the correct expression. On p. 169 *mangaktam* is given for *mangaktanam*; and so on. All this in a cursory way, with growing wonder, not that some errors are made, but that, under the circumstances, they are so few; and that one of a different race, so far removed from the abode of the main body of the people, should be able to produce what so nearly stands the test of intelligent native criticism.

Formerly, when we had time for "systematic" note-book work, we too wrote down a good many things which were "not so"; any one will. But after all, the systematic method must be combined with the other. It *is* systematic, and thoughtful and definite; and definite results, even if mistaken, are good waymarks. On the whole Dr. Seidenadel seems to recognize this, in spite of some apparent assumptions of infallibility.

The author looks askance at missionaries and their language-work; they are so unscientific and take such liberties. Without claiming a right to hold a brief for all missionaries, we may perhaps say that many, even most of them, approach the subject from a very different direction from that of a Chicago University linguist. They want to *use* the language as much as their limited abilities will allow, and if, as in the case of the Bontok dialect, it has never been put down in black and white, they want to commit it to writing, and to printing, and to teach the people to read their own language. Now, as the Doctor constantly affirms, this language is full of variations, transpositions, substitutions; variations in the individual pronunciation, according as a man lives on one side or the other of this river or that mountain. This being so, the missionary's utilitarian object demands that these minor differences be levelled, and that something like a just and intelligible average be struck. In fact he enters into the field of Igorot as a disturbing factor in just the same sense that Caxton did into English. He will doubtless use all the knowledge and good sense he possesses in the task of deciding how he will write words, giving a large consideration to local and antecedent conditions; and then, although he wishes to be humble, he has to seem to arrogate to himself the position of an authority and decide and put into practice what he thinks to be the best.

It is thus that languages live and grow in the presence of civilization and competing languages; they must consolidate and unify, or they degenerate into confused jargons. The Igorot is all the time being hard-pressed by the Ilokano. An immense change has come over the language of the Bontok men who come into contact with outsiders, within the last three years. We missionaries bewail and would fain resist this, as much as the author would. It is a fact to be met, if at all, by training the teachable young to regard their mother-tongue as a dignified, respectable, regular language, not to be dropped at the approach of English or Ilokano, and which, if they will, they can read, and write, and print. If a good nucleus of such intelligent Igorots of the rising generation can be made to see this and take hold of it, the seed will have been sown, we hope, which shall develop into a force to bind language and people together.

Of course this missionary idea is different from the museum, or preserving process, beyond which the linguist, pure and simple, seems unable to think. To bottle up a language in long rows of labeled and slightly varying specimens is very interesting, and of course quite scientific; but it implies that the specimens are, after all, dead things, relics of a past development, but with no special future except for the dissector and microscopist. The two ideas are not antagonistic, unless one or the other assumes the offensive. Let the scientist use his scalpel all he will, so that he understands that he is engaged in vivisection; but let the missionary's dealing with a living race that is begetting sons and daughters that have souls, be also respected.

For the author's grammatical study and analysis of the language we have only words of admiration. Our language helper may find errors which possibly we may be able to recognize; but in general Dr. Seidenadel has gone far more deeply into the construction of the dialect than we have either the time or the ability to do.

His writing of the sounds is different from ours in many respects, and

in some instances we have been at great loss to see how he could justify his system of alphabetization and reproduction of words. One principle which has guided us, that of averages where there is variation, he, from his standpoint would wholly reject, of course. Another important factor is the racial one. His ears and tongue are German; ours are not. Here come in matters beyond dispute. And in writing for people here we have had to take into account the various contacts, Ilokano, Spanish, American.

Taken from the standpoint of the author, the work is altogether and emphatically a most valuable one. It is easily the *opus magnum* on the subject thus far, and will probably remain so until the author can come to the Islands with leisure to cultivate in the home of his friends, of whom he has many, the pursuits of which he has already reaped such noteworthy fruits.

WALTER C. CLAPP.

Postscript.—The foregoing review was written at Bontok, in the Philippine Islands, about a year after the appearance of the book. Early in 1912 the reviewer left the Mission on furlough, and, under a sense of duty and in spite of his affection for the Igorots, decided not to return. Dr. Seidenadel died in Chicago about a year ago. He had long cherished a desire to go to Bontok and study the people and their language there; and it is to be regretted that his patience, enthusiasm and genius are no longer available for that adventure. The tradition of language-work in the Mission, made less imperative from one standpoint by the American development of Bontok as a provincial capital, is not abandoned. Miss Margaret P. Waterman, a graduate of Wellesley College and long recognized as the best practical linguist among the residents in the matter of the Igorot dialect, has completed a study of *Bontok Stems and Their Derivatives* (Bureau of Science, Manila, 1912), and a translation of St. Luke's Gospel (British and Foreign Bible Society) and is now at work on a simple practical manual of grammar. But Dr. Seidenadel's book is likely to remain always a great help to those who pursue the ideal of cultivating the native dialect,—to say nothing of its value as a permanent contribution to the literature of philology and ethnic science.

DANVILLE, PA. W. C. C.

Mr. Albert J. Edmunds of Philadelphia has prepared two postscripts (dated 1912 and 1914) to the fourth edition of his *Buddhist and Christian Gospels* (Philadelphia, 1907-1909). These contain additional notes and bibliographical references, and even sum up Mr. Edmunds's position on the subject of Buddhist-Christian loans at the present time, including an enumeration of the three phrases in the Gospels, which he regards as directly borrowed, and the five narratives colored by Buddhist influence. He has thus formulated a motto which he expects to place at the head of any future articles on the loan hypothesis:

"Both religions independent in the main, but out of eighty-nine chapters in the Gospels, the equivalent of one (mostly in Luke) is colored by a knowledge of Buddhism. The transference was made possible by recently discovered versions of the Buddhist scriptures in vernaculars of the Parthian empire. Parthians were present at the founding of the Christian religion (Acts ii. 9)."

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BENEDETTO CROCE'S ESTHETICS.[1]

NO man who has made the study of literature the real business and the fundamental interest of his life can escape feeling slightly ashamed of the uncertainty and the vagueness of the principles on which his knowledge rests, as compared with the clearness and certitude of the principles of the natural and mathematical sciences. If he limits himself to mere arid learning, his science does not differ much from that of a catalogue or a dictionary; if he attempts, on the other hand, to grasp the spirit of the facts which are the object of his science, he is left to his personal taste without any fixed standard or rational criterion of judgment. In no other subject is it so difficult to grasp both the letter and the spirit, to avoid bare erudition on one side and vain dilettantism on the other.

No such uneasiness seems to have been felt by literary men in the past. This condition of things revealed itself to our consciousness only in the presence of the wonderful development of the natural sciences in the last century; and the remedy for it was sought for, in consequence, in the methods which appeared to be so profitable in that department of human knowledge. Literature, people have seemed to think, has hitherto been nothing else than a kind of harmless folly; let us try to reform it from the bottom up and create a science, a natural science, of literature.

[1] Lecture delivered at the University of Cambridge, November 26, 1913. It is based on Croce's *Estetica*, 3d ed., Bari, 1908, and *Problemi di estetica*, Bari, 1910.

Pure philology assimilated itself to biology, and the history of literature hoped to find at least as firm a basis as that other natural science of human facts, sociology. The arbitrariness had, as by the touch of a magic wand, miraculously disappeared; our feet trod the solid ground of natural certitude ruled by absolute laws.

It was, we must confess, a happy and fruitful delusion, if not perhaps in the study of literature itself, certainly in that of the history of language. The work accomplished was admirable, although its real meaning and value was different from what the people who did it generally thought. But a delusion, no doubt, it was. Now that even biology is realizing that to reduce a fact to its elements is not the same thing as to know it, and that purely mechanical or chemical laws are not sufficient to account for the phenomena of life—now that the natural sciences, in a word, resisting the temptation to put themselves in the place of philosophy, ask philosophy for the concepts which are necessary to make intelligible the study of their data—nobody can any longer believe that the methods which have proved to be inadequate to give us a real, intimate knowledge of natural processes, can succeed in giving a satisfactory explanation of human, spiritual facts.

I do not mean in the least to say that philology has been all this time on a wrong track, but that we have seen only one side of the problems we were studying while we have been nearly blind in regard to the other. We have seen the dead body and not the living spirit; we have constructed the anatomy and not the physiology of language. And the same applies to the so-called scientific study of literature. We have forgotten that the facts which were before us were totally and substantially different from natural facts, and the result is that if we know much more than our fathers did of the external history of these facts we are quite as perplexed as they were when we try to master their

spiritual intimacy, to re-create in ourselves their original life, to exert on them our real human knowledge and judgment. We know a great deal more than our fathers did of the literary events of the past, of their succession and development, but we are not a bit nearer to the great souls of the few poets that really matter. We have advanced, in a word, and enormously advanced, in what the Germans call *Kulturgeschichte*, both from the linguistical and the literary side, but we are always at the same point in the real history of literature. I shall return later on to this distinction, which seems to me fundamental.

The facts of language and literature, we have said, are essentially spiritual facts. Therefore the real remedy for our perplexities does not lie in natural science but in the science of human spirit, that is in modern philosophy. The key is not philology but that much-abused philosophical science, esthetics. But here again we are confronted by many obstacles. First of all, many literary people seem to have a sort of traditional and salutary distrust of all kinds of philosophy; they do not think that they can draw any help from any science which seems to be busy enough with its own internal difficulties and concerning which it is generally affirmed that no decisive consensus of opinion can be found on any point of real importance. This comes from considering philosophy as a science, a particular science, in the great family of sciences, and not, as it really is, as the fundamental activity of every thinking being. We are under a strange delusion when we think that we can do without any philosophy; our thought *is* our philosophy, and none of us can help being a philosopher any more than he can help being, as we shall see, a poet. A student of literature and languages is always, whether he is aware of it or not, an esthetician, and at the root of every error in judgment or method is, either explicitly or implicitly, a false esthetic theory. The thought underlying the great

revival of philological studies can thus be defined as a naturalistic esthetics.

Another delusion, I think, is the absence of consensus in philosophical matters. Philosophy is a free activity—I should like to say *the* free activity of the mind—and it takes as many shapes as there are human minds cooperating in its perennial work. But this variety is not arbitrary, and no real philosopher, no man worthy of the name, can think to-day without accepting the conditions made for his thought by the thought of past ages. Philosophy has its history, and history is inevitably consensus of opinion. All philosophers have always claimed for their systems the merit of being the natural conclusion of all preceding systems. Of course it is not a consensus such as you can find in the natural sciences, where the results can be given without the processes by which they have been reached and are easily intelligible to the profane mind. The consensus of philosophy is such as to require your consent; your own mind must become the last link of the golden chain if you want to realize how contrasting opinions work together in the course of centuries for the truth of to-day.

For there is a truth of to-day which was not the truth of yesterday. Our world is always new to the ever new eyes of the human spirit. In the words of Heraklitus, ὁ ἥλιος νέος ἐφ' ἡμέρηι ἐστίν. Truth is not something that we can fix for ever, an object existing for itself apart from our thought, but the perpetual creation of the human mind, and it is therefore of such an elusive nature as to be really alive only in the always new life of thought itself. Nor are we justified in holding aloof from this work, which is the only conceivable end to the life of humanity as a whole, because we know that our truth shall be superseded by the truth of our sons. On the contrary, it is only by taking the most active part in this work that we can live an effectual life whose action shall still continue in the work of our sons.

But I have gone a little too far from my subject in this refutation of scepticism in philosophy. Let us grant that esthetics is necessary to the study of literature, at least as much as the analysis of the fundamental concepts on which a science is founded is necessary to the understanding of that particular science. But here again the sceptic will find a new justification for his mental attitude in the variety of studies and theories which go under that name. I think I have made it clear that I am talking of esthetics as a philosophical science. The consequence is that all purely psychological esthetics will not satisfy my needs. The object of psychology in this field is only esthetic preference, while the object of philosophy is esthetic activity. And as long as psychology does not interfere with philosophy I believe that its researches may be of great scientific importance, but they do not present any direct interest to the philosopher as such, still less to the critic or to the artist.

Esthetic preference is merely a moment or a particular case, abstracted from its spiritual reality, of esthetic activity; and psychological analysis, however interesting and illuminating in itself, will never be able, without ceasing to be pure psychology and becoming philosophy, to tell us what is the meaning, value and nature of art as a form or grade of the life of the spirit. But when psychologists, forgetting the limits of their science, pretend to give an answer to philosophical problems by using only the abstract concepts and the mechanical methods of psychology, then I believe the philosopher is right if he asks them to mind their own business.

When William James, for instance, places musical pleasure between sentimental love and sea-sickness as phenomena unaccountable by any value for human survival, in fact mysteries if not paradoxes of evolution, we are right in saying that he has turned the problem upside down, because it is not that music must be justified by human sur-

vival, but human survival itself acquires a value and a meaning from the very existence of such men as Palestrina and Beethoven.

The simplest facts of our spiritual life become unintelligible when we see them in the same light as merely natural facts. Nature, indeed, cannot give light to the spirit; the spirit *is* the light of nature, and esthetics as a science is either a part of a complete system of philosophy of the spirit or it is nothing at all. The negative proof of this affirmation is clearly given, I think, by estheticism—if we may so call the body of ideas which gave birth to the esthetic movement. Few men, I believe, though it may seem a paradox, had such a definite idea of the spiritual nature of art as Oscar Wilde; but he had no philosophical training and knew nothing, or very little, of the real thought of his age. So his theories, which would often have found their absolute justification on the higher ground of philosophy, are little more than elegant paradoxes, because he had not a clear consciousness of the nature of his speculation, and kept it on the plane of vulgar thought or common sense, where the paradoxes acquire such an extravagant appearance. He fought vehemently against established prejudices, but the truth he saw could not take in his mind any other form than that of new prejudices which will never, perhaps, establish themselves. To the absurd claims of morality in art he could only answer, "All art is immoral"; but if that is enough, perhaps, to put the problem on the way toward solution, it is by no means a solution, nor even an attempt at a solution. Only philosophy can prove that the two horns of the dilemma are both false, and that, now as ever, *in medio stat virtus*.

II.

The latest development of philosophical esthetics is to be found, I believe, in the works of Benedetto Croce. I re-

gret that the strict limits of this lecture will only allow me to give a very cursory and dogmatic exposition of his ideas, such as must be deemed insufficient by those who are not acquainted with them, and still more by those who are. Only a long course of lectures could give a complete idea of Croce's system, and, esthetics being but a part of it, a knowledge of the whole would in fact be necessary to its full comprehension. On the other hand, Croce's ideas on art and language are of such a nature that they will easily lend themselves to the inference of some conclusions which are of primary importance for our problems and whose truth seems to me absolutely self-evident even from a merely empirical standpoint. This validity and usefulness of its consequences is, to my mind, the best proof of the consistency of Croce's thought.

Croce's esthetics is a science of expression and language. To make clear its character it is necessary to go back to some of its antecedents. The merit of considering art as one of the autonomous forms or grades of spiritual activity belongs to German romantic idealism; but in that period of wild enthusiasm for the newly discovered omnipotence of human spirit art was never able to find its right place in the succession of these forms or grades. Post-Kantian philosophy oscillates between intellectualism, such as Hegel's, and mysticism, such as Schelling's. Kant had prepared this right place in his system, when distinguishing in his *Critique of Pure Reason* transcendental esthetics from transcendental logic; but the pure intuition which was the object of his transcendental esthetics is nothing more than the totality of the *a priori* principles of sensibility limited to the categories of space and time. Art was still for him a mere sensual clothing of an intellectual content.

Croce accepts from Kant the fundamental distinction of esthetics and logic as respectively the sciences of two grades

of theoretical activity of which the second implies the first, but not *vice versa*; and he establishes the same relation in the second part of his system between the two grades of practical activity, the economic and the ethical. This relation is the relation of the individual to the universal. Economic volition is that of the individual, ethical is that of the universal; esthetical or intuitive knowledge is that of the individual, logical or conceptual that of the universal. But what made possible to Croce the identification of this first grade of human knowledge with art was the discovery of the true nature of art made at the beginning of the eighteenth century by Giambattista Vico. Vico can be said to be the real founder of the philosophy of spirit, or idealism, as he had foreseen long before Kant's *Prolegomena* the necessity of the new metaphysics being the metaphysics, as he said, of human ideas. But the whole of his speculation took the shape of an inquiry into the development of human society; he saw "the unity of the human spirit informing and giving life to this world of nations," and his philosophy appeared in the *Scienza Nuova* as an ideal and eternal history of mankind. So it happened that for a long time his work was thought to be essentially philosophy of history or, in the dark days of positivism, sociology. The fact is that his meaning could not be clear except in the light of the great idealistic philosophy; he shared with Bruno and Campanella the function of all Italian thought after the Renaissance, which had been that of foreshadowing and prefiguring the whole development of European philosophy, as by flashing light out of a deep darkness. The diffused light had to come after, by a long, conscious, critical process, generally independent of the work of these pioneers; still, such was the strength and primitive energy of their thought that even after centuries they had some words to say which had not been uttered elsewhere.

"Men," Vico said, "first feel without perceiving, then

they perceive and are perturbed and moved; finally they reflect with pure mind." We have here three grades, of which the first is mere sensation, the limit of spiritual activity; the second is intuition; the third concept. And he went on identifying the second grade—which really is, as we have seen, the first—with poetry. "Poetry," he says, "is the first operation of human mind." Poetry and metaphysics are distinct and opposed: the one is the knowledge of particular things, the other of the universal; the one strengthens the imagination, the other makes it weak; the poets are the senses, the philosophers the intellect of humanity. He described accordingly the first merely poetical society of men, whose symbol was Homer, and the permutation which it underwent when, little by little, the mind grew stronger than the imagination. But this is what we may call Vico's mythology, a part of his thought which is dead and which we must consider as a key to what is still alive. In the description of this mythical primitive society in which he had found the origin of poetry, he discovered the origin of languages also, which he assigned to the same spiritual grade, giving a new and deeper meaning to the Platonic φύσει or natural theory of the origin of language, as opposed to the θέζει or conventional one prevalent in his day.

We are now, I believe, prepared to understand Croce's esthetics; pure intuition, as distinct from and opposed to pure concept, is not the mere sensation which is still formless matter—the limit, as we have seen, of our spiritual activity; it is not perception, or not necessarily and only perception, for perception implies a judgment about the existence of the thing perceived which is immaterial to intuition. Of course experience is the source of all our knowledge, but the knowledge of a certain thing which I do not actually see but only remember, or even only create with my imagination, is an intuition as much as any perception

of external reality. In fact the distinction between the real and the unreal is an intellectual one and belongs to the same class of mental facts to which also belong the categories of space and time; they can be found in intuition, but only *materialiter* and not *formaliter,* as ingredients and not as necessary elements.

But still more important than the distinction between intuition and sensation or perception is the identification of the former with expression. "Every true intuition is at the same time an expression; what cannot objectify itself into an expression is nothing but mere sensation. The spirit does not intuit except by doing, forming and expressing." We must not think only of verbal expressions; there are intuitions which cannot be expressed by words, but only by sounds or lines or colors. But in any case the two terms can be interchanged; what really exists in our spirit is only what we can express. Many mortal men, I know, are convinced of being visited from time to time by the Muse, and believe that what distinguishes them from immortal poets is only the fact that for some reason or other they are not able or willing to express the treasure of poetical feelings that she deposits in their souls. It is a most comforting belief, but I would not advise such people to try and get hold of and put into words or sounds or lines or colors these vague phantasms of their imagination, for it would shatter their comfortable exaltation. They would see them dissolve into air like midnight ghosts at the break of dawn, and in their place they would find a handful of ashes of old lines and half-remembered melodies and half-forgotten pictures, fragments of intuitions which are sufficient to put us in a state not greatly different from that produced by a few glasses of old port, but which have not organized themselves into a new, real, full, effective intuition. Otherwise we would express it, if only in the secret recesses of our hearts, and be the equals of the immortal poets.

The relation is the same between intuition and expression as between volition and action. It is a common saying at least with us Italians, that the road to hell is paved with good intentions, but intentions are not real acts of the will any more than the vague reverie of the dilettante is a real intuition. Every act of the will is an action, and the distinction between the two loses all meaning when we consider, as we do, spirit and nature, the internal and the physical aspect of the same act, not as two entities but only as two different modes of elaborating the only reality, which is spiritual reality. And as therefore there is no action, on the other hand, which is not at the same time a volition, so there is no expression without a corresponding intuition. Every word that we utter has been preceded by its image in our spirit. Language is therefore a perpetual spiritual creation. We are accustomed to seeing dead words and syllables in books and dictionaries, and we believe generally that they are something external, a sort of instrument that we use and accommodate to this or that purpose. But words that grammarians study as independent elements of the linguistical organism are really alive and full of their meaning in their essential function only in the context of speech. The reality of words is only in the spirit that speaks, and every word is new every time that it is employed because it expresses that particular individual moment of spiritual activity which cannot be the same as any other one. Some philologists are inclined to admit that this is true for an original primitive period in which men created language, but maintain that the language so created did develop and does still by association or convention. It is impossible to draw a distinction between the problems of the origin and of the nature of language; and it is only the existence of all the previous expressions which have fixed themselves in the course of centuries and give us the false impression of a body of

language as a reality, independent of the individual activity that has produced the particular expressions, that prevents us from recognizing in the actual linguistical facts the same creative energy that formed the first words uttered by men.

We are not under the same delusion when we talk of the other categories of facts of expression; musical and pictorial language are mere metaphors—and we feel them as metaphors—that help us to collect some characteristics which are common to some of these facts. But in the presence of a certain melody or a certain picture we cannot forget the principle that no expression can give birth to a new expression without first becoming a new impression or intuition.

I know that the whole of this theory, and the identification of intuition with art and language, especially when formulated in the abrupt and imperfect way which alone is possible in a lecture, will raise many doubts and objections. But my aim is and cannot be any other than that of raising such doubts, as they are the best introduction to a more complete study of these problems. I shall try to meet one of these objections, one which probably will present itself to many of you. "What place is given in Croce's esthetics to Beauty? Is not Beauty the supreme object of art? And ought not esthetics to be the science of Beauty?" Beauty is one of those tyrannical words beginning with a capital letter which have kept the thought of man enslaved for centuries. For my part I prefer beauty with a small *b*. How long are we going to suffer this yoke? Or will men learn that words are our servants and that we must master them and make them subserve our ends? Why, I have seen—
d many of you may have seen or heard—the poetry of
t strong and captivating poet, Mr. Masefield, criticised
cause some of the tales told in his poems did not cor-
espond to the ideal of beauty enthroned in the critic's

mind! Of course the same critic will not dare to apply the same criterion to established fames, but he will explain what he would call ugly in Shakespeare if he dared, as having been put in "Macbeth" or "Othello" to enhance by contrast the effectiveness of ideal beauty. Do you not think it would be charitable to explain to such critics that what a poet sings, or a painter paints, is not and cannot be either beautiful or ugly, but that beauty and ugliness are only qualities of the song or of the picture? And so we shall come back to the conception which is in the mind of any sensible man, and this common conception we could express also by saying that beauty in art is only beauty of expression and cannot be anything else. But then you must recognize that beauty and expression are in fact synonymous.

Every man who expresses himself is a poet, quite as much as Monsieur Jourdain, to his great astonishment, *faisait de la prose*. Only we reserve this name for men in whom the esthetic activity manifests itself in a higher degree. But who can draw an absolute distinction between expressions which are art and expressions which are not? Many times in the experience of every one of us we have heard a man deeply moved by his feelings talk in so picturesque and graphic and energetic a language that we felt we were present at the creation of something that even in the stricter sense might have been termed a work of art, not less precious because the words vanished into air, a joy for the instant and not for ever, always to be vaguely remembered and vainly regretted.

And again it will be objected that we do not find beauty only in works of art, or at large in the spiritual creation of men, but in nature as well. An exhaustive answer to this objection would require a full discussion of the relations between spirit and nature as they are seen by philosophical idealism. I have already given before some hints on

this subject; but now I will simply ask any one who has been traveling, I will not say in quite unknown lands but in rather unfrequented spots, whether he has not felt, when discovering beautiful landscapes to which the attention of travelers had never been called before, the same exaltation given by any sort of spiritual activity. He was conscious that this new beauty had not been there before, but that he was creating it by casting on it an artist's eye. The trees, the hills and the mountains, the river and the waterfalls, the green of the meadows and the blue of the sky, are mere sensual *stimuli*; but the beauty of the landscape is as much an esthetic production when seen in nature as when it is admired in a picture. People have looked at the Umbrian landscape, for instance, for centuries, but who *saw* it before Perugino? They have looked at sunsets for thousands of years, but who *saw* them before Turner?

I think some of you may remember that Oscar Wilde wrote what are perhaps his best pages on this subject, although he felt bound to affirm, *pour épater le bourgeois,* that impressionist painting had worked striking changes in the climate of London. The simple truth is that we can to-day enjoy with a keen esthetic pleasure weather that would have been for our fathers nothing more than an awful physical nuisance. The attitude of man toward natural beauty, says Croce in one of the few poetical images that interrupt his clear logical prose, is like that of Narcissus at the spring.

III.

Another objection which has been raised against Croce's theory of art as pure intuition, is that it reduces art to a mere form of knowledge, while what we look for in works of art is the feeling, movement, life and personality of the artist—what we may briefly call the lyrical character inherent in all works of art. The objection is a very

serious one, and if accepted it would shake the foundations of any monistic or idealistic esthetics like Croce's. It would revive the theory of content and form considered, not as they really are as two abstract views of the same fact, but as elements actually concurring with equal power in the production of this fact. The only possible answer, and a very valid one for those who accept the principles of idealism, consists in showing that pure intuition and lyrical character are the same thing, that where the one is the other too will always be found. Pure intuition is knowledge, but not in the mechanical sense given to this word by sensists; it is knowledge as spiritual activity, as creation, and not as mere receptivity or formal association of images. And this sort of activity manifests itself necessarily in lyrical form. What we seek in the works of art is not the empirical personality of the artist, but the δύναμις of his personal esthetic activity, always new and unmistakably his own. We are here very near to Hanslick's famous theory of music as the expression not of any definite feeling, but only of the δύναμις of human feeling. But what Hanslick thought was true for music only is really true for all art. I think it can be said that the true element of art in any work of art is given by rhythm, be it temporal rhythm as in music and poetry, or rhythm of space and form as in painting, sculpture and architecture. We can translate a poem into reasonable prose or into another language, but then the poem is gone. What we call the music of a poem is the poem itself, and our translation will be nothing but a new poem, probably a worse one, suggested by the first. This may seem rather paradoxical at first sight, but then, is there anybody prepared to maintain that the greatness of Dante's "Paradise" arises from his theology rather than from his poetry? We must choose between these two clearly defined positions. Here

again beauty cannot be anything but lyrical beauty, or beauty of expression, which are one thing.

Only, when I talk of rhythm, in poetry for instance, I wish it to be clearly understood that I have not in my mind any prosody or metric. Metrical schemes are pure abstractions, comparable to the idea of species in biology. But the actual life of rhythm is to be found only in this or that line, in this or that definite grouping of vowels and consonants, accents and pauses; we literary people are very apt to forget the real character of the operations of our minds, and we talk of irregular lines whose beauty is derived from a kind of reaction to the ideal scheme of a certain line. It is a fault of the same kind as the one we have seen before when talking of the relation of ideal beauty to art. We take the shadows for living bodies. There are no irregular lines, but only concrete rhythms that we must feel and study, and if a line does not suit the scheme we must remember that the line is always right because it is a reality, and the scheme is always wrong because it is nothing but an abstraction. And I cannot conceive how the beauty of something that has a real existence for itself could be dependent upon its relation with something else that has no existence at all.

This point of view that we criticise in metric or prosody is the same that dominates many so-called literary sciences and gives birth to rhetorical categories and *genres* of literature. Such categories and *genres* are of the same kind as metrical schemes; they are quite legitimate instruments of work as long as we do not forget that there does not exist anything like the idea of tragedy apart from all concrete tragedies, and as long as we do not condemn a new tragedy simply because it is not a bit like the old ones. Every new work of art, far from being bound to obey fixed laws, establishes new laws or, better, has its own law. It must, and will, answer only for itself, and the only claim

that we can put upon it is that of absolute internal coherence.

Again and again, wherever we find rules and types and categories, if we try to get at the heart of things we realize that they are mere shadows and that the only law is that of absolute individuality in art as well as in language. Even phonetic laws are not an exception; we do not obey phonetic laws when speaking, but only the law of the esthetic spirit that makes us find a new expression for every new intuition. It is a common saying in physiology that the function makes the organ and not *vice versa*; and it would be absurd to pretend that the contrary is true when the function is, as in this case, a psychical one. Phonetic laws are merely descriptive summaries of observed facts, and we miss totally the real meaning of the evolution of language when we see it only as a play of mechanical actions and reactions, forgetting the original creative activity of which such actions and reactions are modes and phenomena. In fact I think that many of the criticisms made by men like Driesch and Bergson to some now surpassed biological conceptions, would preserve all their value when applied to that form of pure philology which is represented by the idolatry of phonetic laws. I should like to mention on this subject the work of a German disciple of Croce, Prof. Karl Vossler of Munich, and remind those who may be alarmed by this revolution in philology, that a discussion on the foundations of a certain science need not produce any changes in the body of the science itself. Only, it is always desirable that a man should know what is the nature of the work he is doing.

IV.

If the ideas we have expressed are accepted, the principles a literary critic must always bear in mind are those of the absolute spirituality, individuality and autonomy of

art. These principles, which seem to be three and really are one and the same, will give to literary criticism and to its various branches that unity and organism which for so long a time have been sought for in vain.

As to the first, we must remember the comprehensive formula of one of the greatest predecessors of Croce, Friedrich Schleiermacher: *Das innere Bild ist das eigentliche Kunstwerk,* "the internal image is the real work of art." Those portions of material substance that we call works of art have their only real existence in the spirit who created them and in the spirits who know them by a similar process of creation. Technique is nothing, unless we understand by technical handling of an artistic subject the same artistic production, the succession and progression of intuitions in the artist's mind. No poet can correct a word in his poem, no painter change a line or a shade in his picture if *das innere Bild* has not first spontaneously undergone such corrections or changes in his mind. There is not first a technical standpoint, and after that an esthetical one in the study of art; the painter who learns the first elements of drawing, the poet who exercises himself in the treatment of verse and rhyme, are as yet working in the same sphere of spiritual activity out of which the masterpiece will later on spring forth. And no more are there two different points of view in criticism, but only one: and this consists in the new creation, through the material documents of a former act of life, of the original *innere Bild.* The critic must lend the life of his own spirit to the world that had once existed in the spirit of the poet. And of a poet's world we can really talk, not only in the case of those poets of Æschylean type who see the life of man in a superior sphere of ideal reality, be it a moral or a religious or a merely imaginative one, but for those of Shakespeare's family as well, whose men are the men we meet in this world, because this world is, at least in the poet, an ideal

one and absolutely his own, and no parts of his work can be seen except by the light of his spirit. Reality for itself is blind, but when known it is, and cannot be other than, ideal. This is what I call the critic's respect for the artist's individuality. The critic who reduces the poet's world to terms of his own limited, empirical world, violates and disintegrates it, and will never be able to understand it. In this sense we can say that a good critic must be an objective one, that he must look for the poet behind his work and see the work in the light in which it was born.

To this first and last operation of the critic's mind all sorts of literary study and research must be subordinated. The historical and philological study of literature is the necessary preparation to the critical intuition. But without the former the latter is not even possible. A critical intuition is an historical judgment, and therefore literary criticism and literary history are one thing. Literary history is a history of manifestations of esthetic activity, and many books and researches that go under this name do not belong to it in the least. Let us take the instance of the Elizabethan drama. The question whether Elizabethan drama has sprung from miracle-plays and moralities is a problem of *Kulturgeschichte* and not of literary history. A book like Ch. W. Wallace's on the *Evolution of the English Drama* is only a research into the development of theatrical institutions, and the literary critic may read it to see if perhaps some of the conditions under which the poets of Shakespeare's age worked may have had some influence on the quality of their work. But no history of external institutions and conditions can explain the substance of the work itself, and the same might be said of all researches of sources, of all studies of comparative literature. We are on the threshold of literary history, but literary history is something else. Culture is, in fact, and must be studied

as being, mechanical continuity and relation of times, but art is active originality and creation of a new time.

V.

I wish only to say a few words more on the autonomy of art, because this is the concept that helps us best in distinguishing what is art from what is not. We have seen what are the relations of the esthetical function to the logical one; and everybody knows that a poet who syllogizes is no longer a poet, in the same way that a drama with a thesis is, we know it beforehand, a tedious drama. It is the intrusion of intellectual or moral or practical interests which accounts for failure in art. The sincerity of an artist is not the same as moral sincerity as there is no place in art for truth and falsehood, but only his faithfulness to his pure and real intuitions. This means that intellectual and moral and practical interests can converge in a work of art, but the esthetical activity must completely dominate them and reduce them for its own ends. Otherwise they will be there as dead wood not yet kindled by the creative spark into the flame of life. And the value of the intuition as such does not depend in the least upon the value of the intellectual or moral content: it is not an intellectual or moral value.

There are, of course, many well-meaning people who will never admit that a work of art can be beautiful whose content is immoral. They will tell you that they cannot divide their life into compartments. I shall only observe that there is an enormous difference between the man who leads an animal life and the man who sings the ideal of animal life; the brute has opened his human eyes, and entered into the first light of spirituality. But then I think we do divide our life into compartments; and when I talk to a child, for instance, I am not the same as when I am, unfortunately for my hearers, delivering a lecture. And

yet, I do not think that this means being dishonest. It is simply an operation which is necessary if we want to understand and to be just, and the literary critic must not put the judges out of office. Only we must remember that these compartments are ideal compartments and life remains a deep, indisruptible unity.

No man is empirically a mere poet, no man a mere philosopher; but when we are discussing poetry or philosophy, let us give to Cæsar what is Cæsar's and to God what is God's. And poets are, in the ideal history of mankind, those divine children to whom we know that great reverence is due, and the welcome of a smiling heart.

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THE FUNDAMENTAL LAWS OF ARITHMETIC: PSYCHOLOGICAL LOGIC.[1]

[This article on psychological logic continues the translation of the Preface of Professor Frege's *Grundgesetze,* of which the first part was published in *The Monist* of October, 1915. This part is divided from the first part by the note: "Mathematicians who do not care to study the mistakes of philosophy are recommended to break off here their reading of the Preface."]

FROM the leading current presentations of logic I cannot hope for approval of the distinction that I make between the *characteristic* (*Merkmal*) of a concept and the *property* (*Eigenschaft*) of an object,[2] for these presentations seem to be thoroughly infected by psychology. If we consider instead of things themselves their subjective images—our own notions or presentations (*Vorstellungen*)—all the more delicate distinctions in the things themselves naturally are lost, and others appear instead which are logically quite worthless. And this brings me to what I have to say regarding the factors which prevent my book from having an effect on logicians. It is the injurious invasion of logic by psychology. The conception of logical laws must be the decisive factor in the treatment of logic, and that conception depends upon what we understand by the word "true." It is generally admitted at the very beginning that logical laws must be rules of conduct to guide

[1] [Translated by Johann Stachelroth and Philip E. B. Jourdain.]

[2] In the *Logik* of Benno Erdmann I find no trace of this important distinction.

thought to truth; but this admission is only too easily forgotten. The double meaning of the word "law" is here fatal to clearness of thought with most people. In one sense law says what is, and in the other it dictates what must be. Logical laws can only be called "laws of thought" in the latter sense; they lay down how we must think. Every law that says what is may be understood as dictating that we are to think in conformity with it and is in this sense a law of thought. This holds good in geometry and physics as well as in logic. Logical laws deserve the name "laws of thought" with more right than these other scientific laws only if we wish to express by the name that they are the *most general* laws in that they dictate how we must think whenever we think at all. But the phrase "law of thought" leads people to believe that thinking is governed by these laws in the same way that laws of nature govern events in the world around us. In that case they would be psychological laws, for thinking is a psychical process. But if logic had to deal with psychological laws it would be a part of psychology; and indeed it is sometimes viewed as a part of psychology. These laws of thought are in that case looked upon as rules of conduct in the sense that they indicate the average; just as we may say that healthy digestion takes place in man, or that people speak grammatically, or that people dress in fashion. And then we can only say that what men believe on the average to be true thought, at present and as far as human beings are known to us, is conducted according to these laws. Consequently if we wish to agree with the average man we must think according to these laws. But just as what is modern today will not be modern after some time and even now is not modern with the Chinese, we can set up psychological laws of thought only under limitations of space and time. Such would be the case if logic were concerned with the growth of our opinions on truth and not with truth itself.

These two matters are what psychologizing logicians confuse with one another. Thus Benno Erdmann, in his *Logik*,[3] defines truth as general validity (*Allgemeingültigkeit*), founds this general validity on general certainty of the object about which we judge (*Allgemeingewissheit des Gegenstandes, von dem geurtheilt wird*), and founds this certainty on the general agreement of those who judge. Thus the truth is finally reduced to what the individual supposes to be true. To that I can only reply that the fact of being true (*Wahrsein*) is different from the fact of being regarded as true (*Fürwahrgehaltenwerden*), that it does not matter whether the fact is so regarded by one or by many or by everybody, and that what is true cannot be reduced to it. There is no contradiction in something being true though everybody thinks it to be false. I do not understand by "logical laws" psychological laws of belief, but laws of truth. If it is true that I write this in my room on July 13, 1893, while the wind is howling outside, it remains true even though everybody think it false. If, thus, being true is independent of the truth being acknowledged by some one, then the laws of truth are not psychological laws, but boundary stones on an eternal foundation which may be inundated by our thought but are not movable. And because they are immovable they are important for our thought if it wishes to get at the truth. They do not stand in the same relation to thought as grammatical laws to language; they express the essence of our thought and change with it.

Quite different from this is Erdmann's conception of logical laws. He doubts their absolute and eternal validity, and wishes to limit them to our thought as it is now.[4] "Our thought," I suppose, can only be the thought of human

[3] Halle a. S., 1892, Vol. I, *Logische Elementarlehre*, pp. 272-275. [A second edition of this volume appeared in 1907.]

[4] *Ibid.*, pp. 375 ff.

beings known up to the present time. For there would remain the possibility of discovering human or other beings who could execute sentences contrary to our laws. Suppose that were to happen, then Erdmann would say: "Here is the proof that those fundamental laws do not hold good everywhere." If they are psychological laws, their verbal expression must make known the species of beings whose thought is found by experience to be governed by them. I would say: "There are therefore beings which do not immediately recognize certain truths as we do, but which are perhaps obliged to take the lengthier road of induction." But what if beings were found whose laws of thought were absolutely in opposition to ours and consequently often led in applications to contrary results? The psychological logician could only assent to the fact, and say: "With them the former laws apply and with us the latter." I would say: "Here we have a hitherto unknown kind of madness." Whoever understands by "logical laws" laws which dictate how thought must be guided, i. e., laws of truth and not natural laws of human belief, will ask: "Who is right? Whose laws are in accord with the laws of truth?" The psychological logician cannot ask such a question; for it would mean that he recognized laws of truth which are not psychological. It is hardly possible to falsify the meaning of the word "true" more grossly than by referring to the judge. It is not to the point to object that the remark, "I am hungry," may be true for one and false for another. The remark may be so, but not the thought, for the word "I" in the mouth of another refers to a different being, and therefore the above remark in another person's mouth expresses a different thought. All determinations of place, of time, and so on, belong to the thought whose truth is in question; the truth itself is not subject to place or time. How then does the principle of identity run? Shall we say thus: "In 1893 it is impossible

for human beings to admit that an object may be different from itself"; or thus: "Every object is identical with itself"? The former law treats of human beings and contains a determination of time; in the latter law there is no mention of human beings nor of a time. The latter is a law of truth; the former is one of human belief. Their contents are altogether different, and they are independent of each other, so that one cannot be deduced from the other. It is therefore very confusing to designate both by the same name, "principle of identity." Such mixtures of altogether different things are the cause of the awful confusion which we meet in the doctrines of psychological logicians.

The question why and with what right we recognize a logical law to be true, logic can only answer by reducing it to other logical laws. Where that is not possible, logic can give no answer. Leaving aside logic for a moment, we may say that we are obliged by our nature and outer circumstances to form judgments, and if we judge we cannot reject this law—the law of identity for instance. We must recognize it if we do not wish to bewilder our thought and at last abandon all judgment. I will neither dispute nor try to confirm this opinion, and will merely observe that we have here no logical implication. There is given merely a ground for supposing it to be true and not for its being true. Moreover the fact that we find it impossible to reject the law spoken of does not prevent us from supposing the existence of beings who reject it; but it prevents our supposing that their views are correct with regard to that point. It also prevents our doubting whether we or they are right. That at least is true of myself. If others should dare to accept and to doubt a law in the same breath, they would give me the same impression as if they were trying to jump out of their skins, and I would urgently warn them against such an attempt. He who has once accepted a law of truth

has, by that very fact, accepted a law which dictates how a judgment is to be made, no matter where or when or by whom.

If I review the whole matter it seems to me that different conceptions of the truth are the origin of the controversy. I look upon truth as something objective and independent of the person who judges. It is not so according to the psychological logicians. What Erdmann calls "objective certainty" is only a general acknowledgment proceeding from those who judge, and which therefore is not independent of them but may change with their psychical nature.

We can generalize this still more. I acknowledge an objective domain which is not a domain of actual things; while the psychological logicians, without more ado, look upon the non-actual as subjective. And yet it is impossible to see why that which has a value independent of the person who judges must be actual, that is to say must be capable of a direct or indirect action upon the senses. Such a connection between the ideas is not to be discovered. We can even quote examples which show the contrary. The number 1, for instance, cannot easily be thought actual, unless indeed we are disciples of John Stuart Mill. It is impossible, on the other hand, to assign to each person his number 1; for then we would have to inquire how far the property of these units agrees. And if one person says "once 1 is 1," and another "once 1 is 2," we could only state the difference and say: "Your 1 has one property and mine another." There could be no question of a quarrel as to who was right nor of making an attempt to teach; since there is no common object. Evidently this is quite contrary to the meaning of "1" and the sentence "once 1 is 1." Since the number 1, as being the same for everybody, appears to everybody in the same manner, it can no more be investigated by psychological observations than the

moon can. Whatever notions there may be of the number 1 in different minds, they must be distinguished from the number 1 just as the notions of the moon must be from the moon itself. Because the psychological logicians deny the possibility of the objective non-actual, they suppose concepts (*Begriffe*) to be notions or presentations (*Vorstellungen*) and assign them to psychology. But the weight of truth is too great for this to be easily practicable. And hence comes an indefiniteness in the use of the word "notion;" at times it seems to denote something which belongs to the psychical life of each separate individual and which amalgamates with other notions and associates with them according to psychological laws; and at times it seems to denote something which faces everybody in the same way and in which a person who has a notion of it is neither mentioned nor even tacitly supposed. These two kinds of usage of the word are incompatible with each other; for those associations and amalgamations only occur in particular individuals and with something that belongs to these individuals particularly, such as their joy and pain. We ought never to forget that the notions of different individuals, no matter how much they resemble one another,—and this, by the way, we are unable to ascertain satisfactorily,—do not coincide but are to be distinguished from one another. Every one has notions of his own which are not those of others. Here of course I understand "notion" in a psychological sense. The indeterminate use of this word causes confusion and helps the psychological logicians to hide their weakness.

When will this confusion stop? Everything is finally drawn into psychology; the boundary line between objective and subjective disappears more and more, and even actual objects are treated psychologically as notions. For what is *actual* but a predicate? And what are logical predicates but notions? Thus everything drifts into idealism

and then quite logically into solipsism. If every one denoted something different by the word "moon," namely one of his notions, much in the same way that he would express his pain by the exclamation "oh," the psychological manner of consideration would of course be justified; but a dispute about the properties of the moon would be to no purpose. One person might quite well assert the contrary about his moon to what another person could say with the same right of his. If we could not grasp anything else but what is in ourselves, a conflict of opinions and a mutual understanding would be alike impossible, because there would be no common ground; and such a common ground cannot be formed by a notion in the sense of psychology. There would be no logic capable of being judge in a dispute of opinions. That I may not seem to be fighting against windmills, I will take a definite book and show in it the inevitable sinking of psychological logic into idealism. I choose for that the above-mentioned logic of Erdmann, as it is one of the most recent works of the psychological school and is not likely to be denied all importance. Let us look at the following sentence:[5]

"Thus psychology teaches with certainty that the objects of recollection and imagination as well as those of the notions of morbid hallucinations and illusions are of an ideal nature....Furthermore the whole realm of mathematical notions properly so called, from the series of numbers to the objects of mechanics, are ideal.

In this strange collection the number 10 is actually put on the same level as hallucinations. Here evidently that which is objective and not actual is mixed up with what is subjective. Some objective things are actual, others not. *Actual* is only one of many predicates, and has no more to do with logic than the predicate *algebraic* has to do with a curve. Of course through this confusion Erdmann gets

[5] *Ibid.*, Vol. I, p. 85.

mixed up with metaphysics, however much he tries to keep away from it. I hold it to be a sure sign of error if logic needs metaphysics and psychology, sciences which themselves must have a foundation of logical propositions. Where then is the ultimate basis upon which everything rests? Or is Erdmann's case similar to that of Münchhausen who pulled himself out of the mire by his own hair? I doubt very much the possibility of this Münchhausen-like process even in logic, and suspect that Erdmann will remain in the mire of psychological metaphysics.

There is no objectivity, properly speaking, for Erdmann; for everything is notional with him. Let us convince ourselves of this by his own statements:

"To form a relation between things of which we have notions (*Vorgestelltes*) the judgment needs at least two points of reference between which it takes up its position. As a *statement* about notions (*Vorgestelltes*), it demands that one of these points of reference should be determined as the thing (*Gegenstand*) about which the statement is made, the subject, and the second as the thing which is stated, the predicate."[6]

We see here first of all that both the subject of which something is said, and the predicate, are called objective things (*Gegenstände*) or things of which we have notions (*Vorgestelltes*). Instead of "the thing" might better have been written "the thing of which we have a notion"; for we read: "For things are things of which we have notions."[7] But *vice versa* everything of which we have a notion is also to be a thing: "According to its origin a thing of which we have a notion is either, on the one hand, an object of sense-perception or of self-consciousness, and on the other hand it is either original or derived."[8] What arises from sense-perception or from self-consciousness is certainly of a psychological nature. Things, things of which we have

[6] *Ibid.*, p. 187. [7] *Ibid.*, p. 81. [8] *Ibid.*, p. 38.

notions, and hence subject and predicate, are thus assigned to psychology. This is confirmed by the following passage:

"That of which we have a notion (*das Vorgestellte*) and the notion (*die Vorstellung*) are the same thing: the thing of which we have a notion is the notion, and the notion is the thing of which we have a notion."[9]

The word "notion" is usually taken in a psychological sense; that that is also Erdmann's custom we can see from the following passages: "Consciousness is thus the general concept where corresponding particulars are forming notions and willing";[10] "the forming of notions is composed of notions....and the courses of notions (*Vorstellungsverläufen*)."[11] After this we must not be surprised that an object comes into existence in a psychological way: "As far as a mass of perceptions....offers the same to former excitations and to the stimuli released by them, it *reproduces* the residues of memory which descend from that sameness of former excitations and *amalgamates* with them to form the object of the apperceived notion."[12] On the next page is then shown, as an example, how a steel-engraving of Raphael's Sistine Madonna is made in a psychological way without steel-plate, ink, press, or paper.

After all this there can be no possible doubt but that the object spoken of, the subject, is supposed by Erdmann to be a notion in the psychological sense of the word, as is also the predicate, the thing which is said. If that were so we would never be able to say truthfully of an object that it is green; for there are no green notions. Neither would I be able to say of a subject that it is independent of having a notion formed of it, or of myself as one who forms a notion of it, any more than that my decisions are independent of my will and of myself, the willer; but they would be destroyed with me if I were destroyed. In conse-

[9] *Ibid.*, pp. 147-148.
[10] *Ibid.*, p. 35.
[11] *Ibid.*, p. 36.
[12] *Ibid.*, p. 42.

quence of this there is no objectivity proper for Erdmann, as also results from the fact that he posits the thing of which we have a notion or the notion in general—the *thing* in the most general sense of the word—as the *summum genus.*[13] He is therefore an idealist. If the idealists thought logically they would consider the sentence, "Charlemagne conquered the Saxons," to be neither true nor false, but a fiction, just as we are in the habit of considering, say, the sentence, "Nessus carried Deianeira across the river Euenus." For the sentence, "Nessus did not carry Deianeira over the the river Euenus," could only be true, if the name "Nessus" was borne by somebody. It would not be easy to move the idealist from this point of view. But we need not put up with the falsification of the meaning of the sentence arising from assuming that I wanted to say something about my notions when I spoke about Charlemagne; I simply meant to indicate a man independent of myself and my notions and to make a statement about him. We may grant to the idealists that the attainment of this intention is not quite certain and that perhaps in my attempt I may stray unintentionally from the truth into fiction. But by that nothing can be changed in the sense. By the sentence, "This blade of grass is green," I express nothing about my notions; I indicate none of my notions by the words, "This blade of grass." If I did so the sentence would be false. There now enters a second falsification, i. e., that my notion of green is expressed by my notion of this blade of grass. I repeat: In this sentence there is no question whatever about my notions; that meaning is wholly due to the idealists. By the way, I fail absolutely to understand how a notion of something can be expressed. It would be just such a falsification to say that in the sentence, "The moon is independent of myself and my forming notions," my notion of independence of myself

[13] *Ibid.*, p. 147.

and my forming notions would be expressed of my notion of the moon. With that the objectivity proper would surely be rejected and something quite different put into its place. No doubt it is possible that, in judging, such a play of notions occurs; but that is not the meaning of the sentence. Observe that with the same sentence and the same meaning of the sentence the play of notions may be quite different. And it is this logically indifferent accompanying appearance that our psychological logicians take for the real object of research.

As may be easily understood, the nature of the matter militates against any sinking into the mire of idealism, and Erdmann would not like to admit that for him there is no objectivity proper. But we can just as easily understand the fruitlessness of this effort. For if all subjects and all predicates are notions, and if all thinking is nothing but the producing, connecting, and changing of notions, it is impossible to conceive how on earth we arrive at anything objective. A sign of this vain struggle is the use of the words, "a thing of which we have a notion," and "an objective thing," which at first might seem intended to indicate something objective in opposition to notions. But the words only *seem* to do this, for we have seen that they denote the same thing. Why then this superfluity of expression? That is not difficult to guess. We must notice that an object of our notions is spoken of, though the object is supposed to be itself a notion. Thus it would be a notion of a notion. What relation of notions is meant by this? Though this is obscure, it is quite comprehensible how, from the conflict between the nature of the matter and idealism, such whirlpools can arise. Everywhere here we see the object of which I make a notion for myself confused with the notion, and then we see the difference stand out again. We see this conflict also in the following sentence: "For a notion whose object is general is therefore, as such,

as a process of consciousness, as far from being general as a notion is itself real because its object is posited as real; or as an object which we perceive....to be sweet is given by notions which are themselves sweet."[14] Here the real truth asserts itself. I could almost agree with that. But if we notice that according to Erdmann's principles the objects of which we have notions and the objects which are given by notions are notions themselves, we see that all our straining for agreement must be in vain. I also beg my readers to remember the words "as such" which appear similarly just before the last passage: "Where the actuality of a thing is stated, the subject of this judgment is not the object or the thing as such of which we have a notion, but on the contrary the *transcendental* which is presupposed as the basis of the being of this 'thing of which we have a notion' and is represented in the latter. The transcendental is not to be looked upon as the unknowable....but its transcendence is only to consist in the independence of the process of forming a notion (*Vorgestelltwerden*)."[15]

This is merely another vain attempt to work himself out of the mire. If we take the words seriously, they form a statement that in this case the subject is not a notion. But if this is possible it is not clear why with other predicates which indicate special modes of activity or actuality the subject must be a notion,—for instance in the judgment, "the earth is magnetic." And so we would arrive at the opinion that only in a few judgments the subject is a notion. But if it is once admitted that it is not essential for either the subject or the predicate to be a notion, the foundation is pulled away from underneath the whole of psychological logic. All the psychological considerations of which our books on logic are full just now turn out then to be irrelevant.

But I dare say we must not take the transcendence with

[14] *Ibid.*, p. 86. [15] *Ibid.*, p. 83.

Erdmann quite seriously. I only need to remind him of his declaration:[16] "The *metaphysical* limit of our ideation (*Vorstellens*), the transcendental, is also subordinate to the *summum genus,*" and with that he founders; for his *summum genus* is what we have a notion of, or is notion in general. Or should the above word "transcendental" be used in a different sense? In every case, we are to think, the transcendental is subordinate to the *summum genus.*

Let us reflect a little about the expression "as such." I will suppose that somebody wants to make me imagine that all objects are nothing but pictures on the retina of my eye. Very well, I have no objection to make so far. But now he asserts that the tower is bigger than the window through which I suppose that I am seeing it. To this I would say: "Either the tower and the window are not both pictures on the retina of my eye—in which case the tower may be bigger than the window; or the tower and the window are, as you say, pictures on my retina—in which case the tower is not bigger but smaller than the window." Now he tries to extricate himself from the dilemma by using the words "as such," and says: "The picture of the tower on the retina as such is indeed not bigger than that of the window." At this point I would almost like to jump out of my skin and shout at him: "Well then, the picture of the tower on the retina is not bigger than that of the window, and if the tower were the picture of the tower and the window the picture of the window, then the tower would not be bigger than the window, and if your logic teaches you differently it is not worth anything." This "as such" is an excellent discovery for writers who are not clear in their statements and who do not want to say either yes or no. But I am not going to put up with this hovering between the two, and I ask: "If actuality is predicated of a thing, is the subject of the judgment the notion? An-

[16] *Ibid.*, p. 148.

swer me yes or no." If it is not, then it is, I suppose, the transcendental which is presumed to be the basis of the being of this notion. But this transcendental is itself notional (*Vorgestelltes oder Vorstellung*). Thus we are driven to the supposition that the ideated transcendental is not the subject of the judgment but the transcendental which is presumed as the basis of the being of this ideated transcendental. Thus we would have to keep on continually, but however far we go we never should get out of the subjective. We might begin the same game with the predicate, and not necessarily with the predicate *actual* but, say, with *sweet*. We would then say: "If we speak about the actuality or the sweetness of a thing the predicate is not the ideated actuality or sweetness but the transcendental which is supposed to be the basis of the being of this ideation." But we could not rest there; we would be driven on and on. What can we learn from this? That psychological logic is mistaken if it thinks that the subject and predicate of judgments are notions in the sense of psychology, and that psychological considerations are as out of place in logic as in astronomy or geology. If we wish to get out of the subjective, we have to conceive knowing as an activity which does not create what is known but which grasps what already exists. The illustration of grasping is very well fitted to elucidate the matter. If I grasp a pencil, many different things take place in my body: excitations of the nerves, changes of the tension and the pressure of the muscles, sinews, and bones, and changing of the motion of the blood. But the totality of these processes is neither the pencil nor does it create the pencil. The pencil exists independently of these processes, and it is essential for the fact of grasping that there is something to be grasped; it is not our internal changes which alone make up the grasping. In the same way, what we grasp mentally is independent of those notions and their changes that be-

long to or accompany this grasping. What we grasp is neither the totality of these processes nor is it created by this totality as part of our psychical life.

Let us now see how the finer distinctions in the subject-matter of logic become obliterated in psychological logic. This has already been referred to above when we spoke of *characteristic* and *property*. With this is connected the distinction of *thing* or *object* (*Gegenstand*) and *concept* (*Begriff*) emphasized by myself, and that of concepts of the *first and second stage* (*Stufe*). These distinctions are of course indiscernible to the psychological logician; with such logicians everything is just notion. They have not got the right conception of those judgments which we express by "there is." This existence is confused by Erdmann[17] with actuality, which, as we have seen, is not clearly distinguished from objectivity. Of what things do we assert that it is actual when we say that there are square-roots of 4? Is it 2 or —2? But neither the one nor the other is mentioned here in any way. And if I were to say that the number 2 acts or is active or actual, it would be false and quite different from what I mean by the sentence "there are square-roots of 4." The confusion here under consideration is nearly the grossest possible; for it is not one between concepts of the same stage, but a concept of the first stage is confused with one of the second stage. This is characteristic of the dullness of psychological logic. When we have arrived at a somewhat broader standpoint we may be surprised that such a mistake could be made by a professional logician; but we must have grasped the distinction between concepts of the first and second stages before we can estimate the magnitude of the error spoken of, and psychological logic cannot do that. Here what most stands in the way of psychological logic is that its exponents think such a lot of psychological depth, which is

[17] *Ibid.*, p. 311.

after all nothing but a psychological falsification of logic. And that is how our thick books of logic come to be; they are puffed out with unhealthy psychological fat which conceals all finer forms. Thus a fruitful collaboration of mathematicians and logicians is made impossible. While the mathematician defines objects, concepts, and relations, the psychological logician watches the becoming and changing of notions, and at bottom the defining of the mathematician must appear only foolish to him because it does not reproduce the essence of ideation. He looks in his psychological *camera obscura* and says to the mathematician: "I cannot see anything at all of what you are defining." And the mathematician can only reply: "No wonder, for it is not where you are looking for it."

This may be enough to put my logical standpoint, by way of contrast, into a clearer light. The distance between my point of view and that of psychological logic seems to me so very great that there is no prospect of my having at present any influence through my book upon psychological logic. It seems to me that the tree planted by me would have to lift an enormous weight of stone in order to gain room and light for itself. Nevertheless I would not like to give up all hope that my book may later on help to overthrow psychological logic. As a step toward this end, my book will not, I hope, be quite unnoticed by mathematicians, so that mathematicians will have to come to terms with it. And I believe that I may expect some help from that quarter; for mathematicians have at bottom a common cause with me against the psychological logicians. As soon as mathematicians condescend to occupy themselves seriously with my book, if only to disprove it, I believe I have won. For the whole of the second part is really a test of my logical convictions. It is improbable that such an edifice could be erected upon an unsound base. Those who have other convictions have only to try to erect a similar construction

upon them, and they will soon be convinced that it is not possible or at least is not easy. As a proof of the contrary, I can only admit the production by some one of an actual demonstration that upon other fundamental convictions a better and more durable edifice can be erected, or the demonstration by some one that my premises lead to manifestly false conclusions. But nobody will be able to do that. May my book then, even though it comes rather late, contribute to a revival of logic.

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THE VEDANTIC APPROACH TO REALITY.

PHILOSOPHY is the attempt to think out the presuppositions of experience, to grasp, by means of reason, life or reality as a whole. It seeks to discover a rational explanation for the universe—an explanation which gives to all parts, nature, God and man, their due, views all things in their right proportion and resolves the contradictions of experience. The search for such a solution is the problem of philosophy. The answer should be something in which reason can finally rest. Philosophy has to find out an all-comprehensive and universal concept which itself requires no explanation, while it explains everything else. It must be the ultimate reality into which all else can be resolved and which cannot itself be resolved into anything else. Philosophy is the theory of reality if by reality we mean something that exists of itself and in its own right and not merely as a modification of something else. The test of a philosophical theory is, then, its capacity to coordinate the wealth of apparently disconnected phenomena into an ordered whole, to comprehend and synthesize all aspects of life, reality or experience; for is not the philosopher the spectator of all time and all existence?

Attempts to solve the problem of philosophy generally start from inadequate conceptions which lead us on to more adequate ones through their own inner logic. We start with some part of the whole, some conception which accounts for a portion of our experience, and soon mistake it for the whole or the final explanation of things. We are surprised with contradictions and inconsistencies, which

condemn the theory as an inadequate solution of the riddle of the universe. The mechanical principles of the physical sciences are of great use and value in the region of inanimate nature, but so soon as we apply them to other fields of reality, say animal life, they confess themselves to be bankrupt. Their poverty becomes patent and we, on the basis of these notions and their inadequacies, progress to more concrete and definite theories. Philosophy passes in review the different conceptions which claim to represent the universe, and tests their varying fulness and worth. Philosophy, in this sense, is a criticism of categories. We start with a lower category, criticize it, discard it as incomplete and progress to a higher one where the lower receives its fulfilment. Philosophy, then, is a progressive discovery of reality or defining of reality in terms of fundamental conceptions or categories, or a gradual passage from lower, more abstract and indefinite conceptions, to higher, more concrete and definite ones.

The Vedanta thinkers sometimes approach the problem of philosophy from this standpoint. If we turn to Chapter III of the Taittiriya Upanishad we see there a progressive revelation of the true nature of reality to the seeking mind. The absolute is identified first with one thing, then with another, until we reach a solution which stifles all doubt and satisfies all inquiry by its freedom from discord and contradiction. We here propose to sketch in modern terms the picture of the world as it appeared to those ancient seekers after truth.

The discussion about the nature of reality is in the form of a dialogue between father, Varuna, and son, Bhrigu. The son approaches the father, entreating him to teach him the nature of reality. The father mentions the general characters or the formal aspects of the Absolute known in the Vedanta philosophy as *Brahmam*. It must be something which includes everything else. It is that by which the

whole universe is sustained. "That from whence these beings are born, that by which when born they live, that into which they enter at their death; try to know that. This is *Brahmam*" (Taittiriya Upanishad, Chap. III, 1). The ultimate reality is that in which we live, move and have our being. It is the whole or the totality. "It includes all the world"; naught exists outside it; "there is nothing else beside it"; it is the *res completa,* that which is complete in itself, determined by itself and capable of being explained entirely from itself. Thus the father describes to the son the general features of reality. He gives him the empty formula and asks him to discover by reflection the content of it. The son proceeds to identify it with one thing after another.

The most immediate datum which may be regarded as *given,* and which strikes our mind at first thought, is the world of relatively unorganized matter. One who does not care to strain his thought to go deeper than surface appearances will be struck with the universality and omnipotence of the material forces. Matter is the basis of life. It is the stuff of which the world is made. So the son pitches upon *Anham*[1] (food, matter) as the content possessing the characteristics of the Absolute already set forth. "He perceived that *Anham* is *Brahmam*; for from *Anham* these beings are produced; by *Anham* when born they live; and into *Anham* they enter at their death" (Taittiriya Upanishad, III, 2).

It is the nature of any partial or abstract theory to transcend itself and thus manifest its inadequacy. Matter, though it accounts for a part of experience, cannot be the final explanation of things. Thought can never rest in it. While materialism is a sufficient explanation of the inanimate portion of reality, it does not account for the living

[1] *Anham* is used as equivalent to "matter." See the Vedanta Sutras, II, Adhyaya, III, Pada 1, Sutras 12 and 13. Vidyaranya, referring to a Chandogya passage, says: "Here by *Anham* is meant Earth" or matter.

and conscious aspects of it. If adopted in human affairs it becomes a thoroughly inadequate and false guide. The materialists' picture of the world disregards the specifically human elements of life. The *whole* of experience cannot be identified with this *part* of matter. Our thought rebels against treating parts as wholes. So Bhrigu is convinced that materialism does not effect the unification of reality needed for the Absolute and is therefore not more than a temporary resting place for thought. Dissatisfied with his discovery that matter is the Absolute, he approaches his father for help, and the father asks him to think further. "Desire to know *Brahmam* by reflection" (or deep thought) (Taittiriya Upanishad, III, 2). *Paryalochanam* (reflection) is what the father advises.

The son adopts the advice. Further reflection reveals to him the precise inadequacy of the materialist's theory. In organized matter, the plant world, we come across something to which "matter," though it is the indispensable basis and aid, is not the complete explanation. So this theory of "Matter is *Brahmam*" leaves aside a good deal of the world of existence, while a true theory should cover the whole range of actuality or existence. Mechanical formulas do not account for the life-phenomena. The ultimate reality should be, not matter but something akin to *Prana* (life). "He perceived that *Prana* is *Brahmam*, for from *Prana* these things are born; by *Prana* when born they live; into *Prana* they enter at their death" (Taittiriya Upanishad, III, 3). From this it should not be inferred that the Vedanta philosophy supports a theory of vitalism. That life cannot be completely accounted for on physico-chemical principles is the element of truth exaggerated in theories of vitalism. According to the Vedanta philosophy it is not correct to speak of a sudden revelation of spirit when we come to life, for even matter is spirit, though in its lowest mode of manifestation. It rejects both mechan-

ism and vitalism. We cannot make life mechanical. The world of mechanism is not the same as the world of life. The two are distinct, but the discontinuity between matter and life is not so great as to justify vitalism. The world of mechanism is the medium in which alone life has its being. Though life is not mechanism, still life dwells in it. You find also a tendency to make all mechanism alive. To make life mechanical or mechanism alive is to dissolve the differences in an abstract identity. It would be to sacrifice wealth of content and speciality of service for the sake of symmetry and simplicity. To make mechanism alive would be to deprive matter of its specific function in the universe. Dead mechanism has its own purpose to fulfil, its contribution to make to this wondrous whole. It is therefore not right to reduce unity to identity. We must recognize the difference between the two as much as their unity. The world of matter exists for the purpose of responding to the needs of life. The name *Anham* (food) is advisedly given by the Vedanta philosophers to the principle of matter. Matter exists for the purpose of being *used up* by life. It serves as food for living beings. It is not an alien element, but is something which can be "eaten," controlled and utilized. It is the food which enters into the organic life, the material which the organism uses to build up its body. The authors of the Upanishads make it clear to us that environment, with its necessity, is not a recalcitrant force, not some dark fate over against which we have to knock our heads in despair, but rather the servant of the organism, the helpmate of life and consciousness enabling the growth and perfection of higher beings. In short, life and matter, organism and environment are members existing for each other in a larger whole. They are unintelligible when viewed in separation. "Matter is rooted in life and life in matter" (Taittiriya Upanishad, II, 3). The science of physics, which seeks to divorce matter from life and study

matter in its isolation, studies an abstraction, however useful it may be. The ideal of physical science is an explanation of life in terms of mechanism. Anything which comes in the way of this mechanical ideal is quite unwelcome to physics. Again, if the science of biology concerned itself with life to the exclusion of matter, it would be a science of dead abstractions. What we need is biophysics and physicobiology; they only would do justice to the different aspects and their essential unity. The whole must be seen as a whole if it is to be seen at all. We see then the exact relation of life and matter. The same whole of reality manifests itself first as matter, then as life. The two are but lower and higher expressions of the deeper reality. They are but movements in one grand scheme. Life, being a higher stage than matter, is the completer truth. Life is the promise and potency of matter. Life is the soul and spirit of matter. The Upanishad says of matter that "this *Prana* (life) produced in the body is the soul." So life includes and transcends matter. It is a higher concrete than matter. Matter is a fragmentary abstraction from the point of view of life. The mere externality of matter is transcended and overcome. The parts are no more external to each other but they are elements in an organic whole with a definite end. In the living body the elements cooperate in the preservation of the organism. But even in the living body there is an element of externality which will disappear as we proceed to the next higher category of *Manas* (mind) or consciousness.

The whole world of reality refuses to be squeezed into the category of life. Though *Prana* or life is nearer to reality than matter or mechanism, still it cannot account for the whole of our experience. Life, for instance, cannot account for consciousness. The category of life, failing to embrace the whole of reality, confesses itself to be but a partial truth covering only a limited field of experience.

It cannot therefore be put forward as the ultimate essence or principle of the whole world of reality. Once again the son approaches the father. The father asks him to think to the bitter end without stopping at halfway houses. He pursues his reflection and discovers that the higher forms of life require us to introduce another category to describe their relations. The new factor of consciousness makes its appearance as life develops. *Manas* or perceptual consciousness is the sole reality. "He perceived that *Manas* is *Brahmam,* for from *Manas* these beings are born; by *Manas* when born they live; into *Manas* they enter at their death" (III, 4). Here by *Manas* is meant perceptual consciousness which delights in sense objects and is moved by instincts and impulses.

The relation of mind to life is exactly of the same kind as the relation of life to matter. "Mind is the soul of *Prana* or life." Mind is not a by-product of body or life but is the central core of life. The two are different expressions of the one spiritual essence, lower and higher stages of a single all-embracing life. The relation of mind to life is that of a higher to a lower aspect of the spirit. It is puerile to minimize the distinction between the two by materializing mind or spiritualizing matter and life. While recognizing the distinction we should not lose our grip on the essential unity which underlies the distinction. The two contribute in their own distinct ways to the same individual whole. The two are so fashioned and constructed as to develop and promote a complete identity. They are aspects of the ultimate spirit, through the interaction of which the whole realizes itself. The science of biology, which studies life, neglecting the fruit and essence of life, mind, studies an abstraction. Psychology, if it divorces mind from life and studies mind as an isolated phenomenon, apart from its setting of life and the organism, lays itself open to the fallacy of the abstract.

It studies not human minds but disembodied ghosts. It is "phantomology" and not psychology. It is a good sign that psychology at the present day views its subject-matter from the biological point of view. Psychology studies not merely the *psyche* but the psychophysical organism. The conscious organism can be seen as a whole only by bio-psychology or psychobiology. Only then shall we know mind in its origin and working.

The concept of *Manas* (mind) is higher than life or matter. It is the richer, fuller and more inclusive concept. But the searching intellect is not satisfied with its adequacy, for the perceptual consciousness does not exhaust the nature of reality. No doubt it accounts for the animal mind. Animals have only a perceptual consciousness, their mental horizon being restricted to mere perceptions of the present moment. The animal lives only in the present. It is devoid of the power of synthesis and therefore of self-consciousness. But the human consciousness is capable of rising above itself, of comparing itself with other selves and of passing judgment on its own character. The man judges while the animal only senses. He is a being of "wise discourse looking before and after." He is able to transcend the animal limitations, break down the despotism of the senses and lift himself above himself. While the animal leads a life of mere feeling and impulse, the self-conscious individual regulates his life in conformity with ideals of beauty, goodness and truth. It is the capacity to distinguish fact from idea which makes possible art, morality and science. So a higher category than animal mind or perceptual consciousness is felt to be needed. He approaches his father and is advised by him to think to the root of the matter. The son realizes, on reflection, that the specific quality of man which makes him the lord of creation is his *intellectuality*. By his intellect or understanding he seeks the true, attempts the good and loves the beautiful.

By it he connects sensations, compares and contrasts them with one another and derives inferences. It gives the power of synthesis. To it is due the self-consciousness of man. So the seeker after truth hits upon *Vignana* or understanding. "He perceived that *Vignana* (intellect) was *Brahmam,* for from *Vignana* these beings are born; by *Vignana* when born they live; into *Vignana* they enter at their death" (III, 5).

What is the relation of *Vignana* to *Manas,* or understanding to perception? This is the familiar question of modern epistemology, the relation of the universal to the particular, concept to percept, thought to sense. Understanding is related to perception as perception to life, or as life to matter. *Vignana* is a higher form of the lower *Manas.* It is the soul of *Manas* or its essential reality. "*Vignana* is the soul (or spirit) of *Manas*" (II, 4). Nothing is gained by divorcing intellect from sense. Such a divorce leads to abstract explanations of reality. Sense is the condition of thought. Thought does not produce or create a new order of existence. The sense world is not a mere chaos of particulars into which thought introduces, later and from outside, order and system. Thought only discovers or explicates the order which already prevails in the world of facts. The ideals of the world reveal themselves to thought. We seek order *of* facts. As in science we try to interpret the order prevalent in the actual and discriminate it from our errors and prejudices, so in morality we try to see the goodness of things and discriminate the good from the bad. We are not creating a new moral world by our action. The tendency to neglect the perceptual basis is the besetting temptation of the intellectualist temper. Rationalist theories which sacrifice the particular to exalt the universal reduce the universe, in the vivid phrase of Bradley, to an "unearthly ballet of bloodless categories." We get a philosophy of arid concepts having nothing to do

with the glowing experiences of life. Truth becomes a dead conformity to certain logical conceptions and ideas with no promptings from life. In art technique gets the mastery over temperament. Art expresses the critical and not the creative attitude of life. Morality becomes the drill-sergeant type, insisting on nothing more than a blind unthinking obedience to the commands delivered. Rationalism thus murders reality to dissect it. We find a mechanical perfection in place of spiritual beauty, logic in place of life. Organization is the ideal, but the process of starving the real leaves no material to organize. Philosophy becomes arid and abstract, art mechanical and soulless, and ethics formal and dead. The dire consequences resulting from the adoption of this theory in practical affairs of the world, we see to-day on the fields of Europe. We find also systems of philosophy which protest against this deification of intellect. But in their righteous revolt against the abuse of logic they are led to the opposite extreme of advocating inordinately the claims of immediate experience. Bergson and James are representatives of this new tendency in philosophy, which goes by the name of intuitionism or radical empiricism. This tendency to exclude logic from life is as vicous as the other tendency to exclude life from logic. The abstract and one-sided nature of mere empiricism is reflected in the world of philosophy, art and morality. Under its influence the superficial aspects of things are noted and the underlying principles neglected. Naturalistic explanations become dominant in philosophy. Art is sensualistic and ethics economic or utilitarian in the lowest sense of the term. Mere percept and mere concept are both good for nothing. Both are abstracts reified. Kant spoke a great truth which the world cannot afford to forget when he said that "percepts without concepts are blind; concepts without percepts are empty." This essential unity of these two distinct factors the Vedanta thinkers recognize.

The self-conscious individual in whom *Vignana* functions at its best becomes the highest expression of reality if there is nothing higher than intellect. But self-consciousness which is the product of intellect presupposes self-distinction. At the intellectual level the self conscious of itself is self exclusive of others, one among many. The self not only distinguishes itself from others but excludes others from its nature. A "pluralistic universe" will be the last word of philosophy, but the thinking mind recognizes certain difficulties in the way of accepting this solution as final. The natural outcome of such an intellectualist pluralism will be a narrow philistine spirit of individualism, sensualism and selfishness. The individuals enter into rivalry with one another for the satisfaction of their appetites and ambitions. Such a view will develop a sort of morbid ease and self-satisfaction with the actual and thus curb all efforts for the improvement of mankind. It would make it impossible for the finite mind to transcend its finiteness. It gives man no ideal of the solidarity of the universe to which he has to work himself up. The human consciousness which in some moments of exaltation feels itself to be at one with the whole universe, baffles this intellectual analysis. Those aspects of experience known as religious are not accounted for by the pluralist scheme. The factor of ever aspiring, ever striving for something higher which man has not but hopes to have, is not satisfactorily explained. Man recognizes his incompleteness and imperfection and seeks for something above himself, an ideal, an infinite. If the individual's highest aim is merely to secure an independent status for himself he becomes divorced from his real, i. e., his divine self. It is impossible for man, a child of eternity, to distinguish himself from God in the long run. He cannot fix any boundary to his real self. If he seeks for the private self-satisfaction he seeks the finite as if it were the infinite. It is the self-contradiction of a

being who knows not what he really is and seeks his good where it can never be found. If the world is a number of distinct isolated units, then peace and harmony are *a priori* impossible. Pluralism by itself cannot give any satisfactory account of the unity of the world of spirits. Most of the modern pluralistic systems recognize this difficulty. Professor Ward says: "That a plurality of individuals in isolation should ever come into relation is inconceivable indeed, but only because a plurality without unity is itself inconceivable" (*Realm of Ends*).

There is no doubt that human self-consciousness represents, though not the highest, yet a very high manifestation of reality. Sankara gives the following statement: "The *Atman* is expanded only in man. He is most endowed with intelligence. He speaks what is known, he sees what is known. He knows what is to come, he sees the visible and the invisible worlds. He desires to obtain immortality by appropriate means. Thus endowed is man." He has ideals of knowledge, beauty and goodness but he does not as a finite consciousness realize his aspirations. He only struggles toward union, peace and harmony. Though he ever strives toward union with the whole or the divine, he never grasps it on account of his finiteness and impotence. Finite souls never realize, though they ever strain after, that pure bliss and self-forgetful realization which in Vedantic phraseology is called *Ananda*. The sciences belonging to the intellectual level are sciences of struggle and endeavor and not sciences of fruition or fulness of attainment. They are sciences of approach to reality. Logic with its impulse toward totality *demands* a complete and consistent world; love *struggles* for union with the whole, and life *attempts* to realize the all-perfect in conduct. In all these regions of mind we catch glimpses of the real but do not have the full vision with its joy unspeakable and the peace that passeth all understanding. We have *demands,*

struggles and *attempts*. We are in the *striving stage*. We are only on the road with a dim vision of the end; the fulfilment is still a distant scene. The full splendor is not yet. So human self-consciousness is incomplete and imperfect. It is only a grade of reality to be transcended in something higher but not the whole of reality. On the other hand, if intellect should be the highest phase of reality, then morality, law and justice become the ultimate terms and struggle the end of existence. What a poor imperfect thing man will be if he has no prospect of realizing his ideals! His effort to become something greater, holier and higher than his own finiteness will be unsuccessful. The world will be cut into two as with a hatchet, self and not-self. If we do not embrace them in a final higher unity, then his spiritual endeavors are foredoomed to failure. Pessimism is our only refuge and prayer all our business. Man presses on toward a higher life, but cruel fate crushes the human soul. He desires to throw off his brutish heritage and reach heaven. But the blind forces of nature which go on their relentless way caring naught for the human victims, dash him down to the bottomless void. The intellect with its vision confined to outward appearances, is struck with "nature red in tooth and claw." Such an outward vision gives the impression that we are caught in the wheels of a soulless engine which has neither the eyes to see our agony nor the heart to feel for us. We are the victims of a merciless fate, trapped in the grip of destruction. Intellectualistic despair is the mental attitude of those who break the real into self and not-self and make the universe a tug of war between the two. Matthew Arnold's insistent note of sadness is due to his theory of opposition of self and not-self.

"No, we are strangers here, the world is from of old. To tunes we did not call, our being must keep chime." The system of nature does not sympathize with the bliss for

which we sigh. Our boundless hopes are shattered to dust and our tenderest ideals mocked by the stern indifference of nature. The microcosm is pitted against the macrocosm and to all outward appearances the external world seems to be the more potent force. What can man do in this plight except withdraw from the world and obtain inner freedom by renunciation and contemplation? "By the Tiber as by the Ganges, ethical man admits that the cosmos is too strong for him, and, destroying every bond which ties him to it by ascetic discipline, he seeks salvation in absolute renunciation" (Huxley, *Romanes Lecture*, p. 29). The Sankhya philosophy of ancient India starts with a dualism of *Purusha* (self) and *Prakriti* (not-self). They are the two eternal uncreated substances differing essentially from each other. Deliverance is to be obtained by realizing the separateness of the two and dissolving the bond between them. Man to gain his freedom has to cut himself off from the ties that bind him to nature. We are exhorted by Mr. Russell in his admirable essay on the Freeman's worship to cherish, adore and love the ideals where the mind is at home, caring naught for the universe. He builds an ethics of renunciation on this "firm foundation of despair." "To abandon the struggle for private happiness, to expel all eagerness of temporary desire, to burn with passion for eternal things, this is renunciation and this is the Freeman's worship." We are engaged in an unequal struggle between man and nature, self and not-self. A mere contemplation of it would produce a stoic calm combined with a stern pathos.[2] Militant heroism we may adopt if we care

[2] "Nature is cruel, man is sick of blood;
Nature is stubborn, man would fain adore;
Nature is fickle, man hath need of rest;
Nature forgives no debt, and fears no grave;
Man would be mild and with safe conscience blest;
Man must begin, know this, where nature ends;
Nature and man can never be fast friends.
Fool, if thou canst not pass her, rest her slave."—M. Arnold.

How pathetic is this expression of despair, born of an intellectual vision which disdains to dive beneath appearances (Cf. Russell, *Philos. Essays*).

for the martyr's crown. Even martyrs die with the complaint, O God, why hast thou forsaken me? The destiny of man seems to be struggle, unrest, and baffled hope. This pessimistic conclusion is the essential theme of the Buddhists. They say there is nothing else than this world process or *Samsara*. There is neither a changeless God responsible for it nor a suffering deity struggling against the attacks of Satan. Buddhism considers the appearance of opposition to be final and exhorts man to get out of this whirlpool by sinking his selfhood. But this is too harsh a conclusion to be accepted by all. So a supreme soul or *Iswara* soon appears to help the individual in his warfare against the not-self. So God along with man battles with the prince of darkness. The atheism of the Sankhya system gives place to the theism of the Yoga philosophy. We have then the individual self, God and nature; the individual *self*, according to Saiva Siddhanta, *Vaishnavism* and Christianity, has to extricate himself from the fetters of *Nature* by the grace of *God*. The Highest in all these theistic systems is looked upon as a personal godhead,—a father, creator or providence, accessible to prayer and propitiation, ever loving man and granting his requests. By the help of God it is possible for man to escape out of this drift of the world called *Samsara*. If we think in the acquired dialect of the intellect we will not be able to reach the highest which includes all other things. We will get a pluralistic universe presided over by a God whose position therein is ambiguous. If we say God is over against a number of spirits and that the Absolute is a republic of spirits including God, we ask, what is the position of God in the republic? If he is one among the many he is reduced to the level of the finite beings. If man himself is part of God we shirk the whole problem by raising man to the level of the infinite. Pluralism is displaced by an abstract monism. But the pluralists' God is not the perfection

transcending both good and evil, not the absolute which absorbs them both, but only a force within it fighting with another. Such a God can only be an aspect of reality and not the whole of it. Besides, this conception of God opposed to the world naturally culminates in deism. God is transcendent to the world because the world is evil and he is good. He has nothing which nature has and can only be defined negatively. So a severe logician of the type of Sankara who thinks to the very foundations, with his intellectualist bias, reduces the universe to an opposition of self and not-self, God and the world, the infinite and the finite. Certainly both cannot be real, for the two are exclusive of each other. The finite world is dismissed as illusory and the absolute posited as real. For if we argue about the problem of the origin of the world and man's place in it, we will be drowned in a sea of contradictions. Kant, and after him Bradley, have shown the difficulty of reconciling the antinomies with which our understanding confronts us. The self-contradictory cannot be real. Therefore the finite world is illusory and the Absolute is real, for it is pure affirmation. But the Absolute which repels the relative cannot be anything more than an undifferenced unity which is the negation of the finite and the determinate. The Absolute is related if we can talk of relation in this sense, only negatively to the world. The Absolute thus collapses into a self-identity, negatively related to the particulars, a featureless unity leaving aside all differences. To this absolute none of the attributes of finite being belongs. If we attach any predicate to it we will bring it down to the level of the finite. It is not anything which the finite world is. If the finite world is many it is one; if it is complex it is simple; if it is varying it is constant; if it is temporal it is eternal. Strip off everything finite and what remains is the infinite or God. Everything positive is excluded from the real, mind and matter in-

cluded. Escape from finite life is the goal of humanity. Such are the views of Sankara and the neo-Platonists. The fatal criticism against all such abstract notions of the Absolute is that they do not give any explanation of the finite universe. To say that the Absolute is the external and accidental cause of the universe, is no answer. To dismiss the world as illusion only removes the difficulty a little farther, for the question still arises, What is the cause of this world illusion? Thus we see that if we stick fast to the intellectual level we have either a bare unity as in Sankara or a collection of separate elements as in Sankhya and the Yoga. But in no case is it possible for us to have a unity in diversity, an organic system in which the whole should be known through the distinction and relation of all the parts. We do not see the two, unity and diversity, as elements in a whole or factors in a unity. It is such a solution that is adopted by the Vedanta philosophers.

The distinction between self and not-self is not an irrational surd which cannot be eliminated, but is a distinction within a unity. In man there is a struggle between the higher and the lower, self and not-self (*Purusha* and *Prakriti*). He is an amphibious animal living in two worlds. Born of matter, entangled in it and oppressed by want and misery, he still has the divine spark which gives him a place in the spiritual realm of freedom. But the struggle between the divine and the human is bound to result in a complete triumph of the spirit and the consequent idealization of the material aspect. The self with its "ought" comes down on the not-self and, in spite of the refractory nature of the latter, transforms it. In morality we transform the actual and idealize it. Knowledge presupposes a unity between subject and object; without this basis knowledge is impossible. The very distinctions made by the intellect presuppose a unity which is not grasped by intellect. The interpretability of nature is proof posi-

tive of the kinship of object with subject, nature with mind. The antithesis between self and not-self is resolved in the Vedanta philosophy and the two are reconciled; "*Purusha* (the self) is the eater, *Prakriti* (not-self) is the food, and, abiding with it, he feeds" (Maitrayana Brahmana Upanishad, VI, Prapathaka 10). The not-self offers the conditions which are the material of self and the self instead of being the slave of the not-self is the highest and the most articulate expression of the not-self. Self and not-self do not run counter to each other. They are no rivals; rather do the two help each other in fulfilling the mission of the Divine. They are co-operating and not conflicting elements in the whole. We cut in two the whole and then view the environment as an alien influence checkmating the individual at every step of his progress. The individual is said to progress by fighting and conquering nature. We forget how nature could not be conquered by him if it were different from him in its essence. It is therefore a system of absolute idealism, however much we may try to disguise it by giving it other names that preserve to us the reality of the ideals and the unity of the pluralistic world. Even thinkers strongly inclined to pluralistic notions are compelled by sheer force of logic to embrace their pluralism in a higher idealism. Upton says: "It follows therefore that, though atoms and bodies appear to be isolated co-existences in space, this complete isolation and seeming independence of each other is only an appearance; for the reciprocal causality by which all these atoms and bodies are linked together inevitably forces us to the conclusion that deeper than the apparent spatial distance and division there is a metaphysical unity, or in other words that the self-subsistent creative ground of all finite existence does not wholly separate Himself from any one of the plurality of dependent energies or beings into which He differentiates himself; and therefore as every finite atom or

finite soul still remains, as regards a part of its nature, in indivisible union with its self-subsistent ground and source, the common relation to the self-subsistent one affords a true explanation of the metaphysical unity of the cosmos, and also of the possibility of reciprocal action of the monads of nature on each other, and of reciprocal action of the finite mind on nature and of nature on the mind. Thus the most recent science and philosophy appear to assert at once a real pluralism or individualism in the world of finite beings, but at the same time a deeper monism. The Eternal, who differentiates His own self-subsistent energy into the infinite variety of finite existences, is still immanent and living in every one of these different modes of being, and it is because all finite or created beings are only partially individual and still remain in vital union with their common ground, that it becomes possible for them through the medium of this common ground to act dynamically on each other; and it is for the same reason that those finite beings such as man, who have attained to self-consciousness, are able to enter into intellectual, moral, and spiritual relations, both with other rational finite minds and also with the eternal being with whom their own existence is in some measure indivisibly conjoined" (*Bases of Religious Belief,* pp. 12-13). The latest and the ablest exponent of pluralism, Dr. Ward, says: "Faith in God as the ground of the world affords us an assurance which we could not otherwise have, that complete harmony and unity, the good of all in the good of each is really attainable, nay will verily be attained. Whereas if we stop at a plurality of finite selves in interaction, we have no guarantee, cannot even reasonably expect that such a totality will ever attain to perfect organic unity" (*The Realm of Ends,* p. 447). Thus Ward and Upton, no friends of absolute idealism, are driven to admit the existence of an all-embracing unity as the ground of the world and recognize the finite selves

as differentiations thereof, though they try very hard to give the finite souls separate individualities.

The reality of the ideals of knowledge, art and morality has for its basis the highest unity which cannot be realized by *Vignana* (intellect) which revels in distinctions of self and not-self, subject and object, man and the universe, organism and environment. Our knowledge aspires to something more than knowledge, an intuitive grasp of the fundamental unity; our morality to something more than morality, viz., religion; our self to something more than personality, viz., God or the Absolute. Our knowledge is incapable of bringing us into contact with the whole. It aims at the unity, though the limitations of intellect forbid the attainment of the unity. The highest unity "from which all speech with the mind turns away, unable to reach it" (Taittiriya Upanishad, II, 4) cannot be grasped by the intellect.[3] The universe does not spell out its secret to man. It withholds from man the mystery which he strains to see. The human understanding can classify, relate and create out of given data, but it cannot say anything about the Absolute which is one without a second, and which is no object of the senses but constitutes the self of the whole world. The Kena Upanishad says: "It is other than the known and above the unknown." Simply because it is not open to knowledge we cannot say it is unreal. The illusions and contradictions of the intellect according to the Vedanta philosophies only exhibit the insufficiency of intellect to grasp the whole. They only show that there is a higher form of experience and that the spiritual life is not exhausted by the intellectual. To realize that there is the one all-encompassing reality including self and not-self, we have to proceed to the next higher stage. Finding the

[3] Kena Upanishad says: "The eye does not go thither, nor speech, nor mind. We do not know, we do not understand, how any one can teach it. It is different from the known, it is also above the unknown" (I, 3-4. See also I, 5-9).

finite intellect infected with duality, and realizing its inadequacy to represent the real, the son approaches the father, who asks him to persist in his inquiry. Bliss (or *Ananda*) reveals itself as the final explanation. "He perceived that *Ananda* is *Brahman;* for from *Ananda* these beings are born; by *Ananda* when born they live; into *Ananda* they enter at their death" (III, 6). We have direct experience of this bliss or delight in philosophic contemplation, artistic worship and religious devotion. In them we gain the ultimate peace beyond the unrest of life, attain the glorious harmony transcending all discords and grasp the unity of purpose which works through the apparent conflict of natural and social forces. The seer, the sage and the saint all enter into direct communion with the heart of things. Self and not-self are felt to be clasped in one in that stage. "All fears cease." Incidents of the earth cease to trouble the knower. The self has the consciousness that there is nothing else beside the Absolute. "One finds nothing else, knows nothing else, but the self." "All this is the self and the self alone" (Brihadavanyaka Upanishad, II, 4-6). So long as he sticks fast to the hard distinction between self and not-self, he has not reached the highest. It is said, "Where one sees nothing else, hears nothing else, understands nothing else, that is the infinite. Where one sees something else, hears something else, understands something else, that is the finite" (Chandogya Upanishad, VII, Prapathaka, 24th Khanda). The oneness of the universe cannot be characterized by anything else than bliss, joy or delight. "Seeing the self by the self, he is satisfied in his own self" (*Bhagavat Gita,* VI, 20). This highest experience is the heaven of Dante, free from darkness, confusion and antagonism. It is characterized by peace, perfection and tranquility. The aspirations of knowledge, love, morality, are here transformed into actualities. The unity of subject and object is no more an

ideal but we see it face to face. The oppositions of the finite consciousness are all reconciled. The son arrives at this stage and is no more troubled with doubts. His inquiry ceases. From *Ananda*, matter, life, consciousness and understanding are born, in *Ananda* they live and to *Ananda* they return. The harmony of man and the universe, *chit* (intelligence) and *sat* (reality) is realized. In that moment of divine vision described in the Bhagavat Gita the whole choir of heaven and furniture of earth was seen by Arjuna moving in the radiance of God. This religious or intellectual experience is the summit of the whole evolution. It is the crowning round of human life. It is the completion and the consecration of the whole struggle. It is "the light that never was on sea or land, the consecration, and the poet's dream." Here the philosopher's quest for reality in which thought can rest, terminates.

If self-consciousness is the distinctive mark of the intellectual experience, self-forgetfulness characterizes the *Ananda* (bliss) condition. It is the state where the self loses itself in the universe and by so losing finds its own realization. Peace and harmony we have; for the self offers itself up wholly and completely to the service of the Absolute. So long as we feel ourselves to have individualities of our own, we will be beset with conflict and contradiction, pain and pleasure, but when once we disinterestedly give ourselves up to the whole, there is an end of all discord. "Whatever thou doest, whatever thou eatest, whatever thou sacrificest, whatever thou givest, in whatever austerity thou engagest, do it as an offering to me" (*Bhagavat Gita,* IX, 27). "Fix thy mind on Me, be devoted to Me, sacrifice to Me, bow down to Me. Thus steadied, with Me as thy Supreme Goal, thou shalt reach Myself the Self" (*Bhagavat Gita,* IX, 34). Only this complete renunciation of self and delivering up to the whole, will liberate

us from the pains of opposites (cf. *Bhagavat Gita,* IX, 28). The beautiful tradition that no man can see God and live, points to this truth that finite selfhood is incompatible with the life of the spirit. It shows how we cannot see God until we roll the stone of self away. The religious individual feels himself to be, not a selfish atom in the universe, but part of an order with a station to occupy and a function to fulfil in the economy of things. With his vision ever on the supreme, the religious soul approaches the facts of existence. He knows that the forces of the world cooperate with him in the realization of the highest. He lives above the plane of human experience, but still in it. He is the hero of the world who deserves worship at our hands.

It is not right to presume that intuition, by which we see the oneness of things, negates whatever intelligence posits. Intuition is really the soul of intelligence. The unity we will be able to grasp by means of intuitive insight, is the presupposition of all intellectual progress. Intuition is only the higher stage of intelligence, intelligence rid of its separatist and discursive tendencies. While it liberates us from the prejudices of the understanding, it carries our intellectual conclusions to a deeper synthesis. Instead of being an unnatural or a mysterious process it is a deeper experience which, by supplementing our narrow intellectual vision, amplifies it. Intuition is not an appeal to the subjective whims of the individual or a dogmatic faculty of conscience or the uncritical morbid views of a psychopath. It is the most complete experience we can possibly have. It is the experience devout souls have in moments of spiritual exaltation or religious devotion. Hegel, and after him Bradley, testify to the highest worth of this religious experience. Hegel says: "All the various peoples feel that it is in the religious consciousness they possess truth, and they have always regarded religion as constituting their true

dignity and the Sabbath of their lives. Whatever awakens in us doubt and fear, all sorrow, all care,—we leave behind on the shores of time; and as from the highest peak of a mountain, far away from all definite view of what is earthly, we look down calmly on all the temptations of the landscape and of the world, so with the spiritual eye man, lifted out of the hard realities of the actual world, contemplates it as something having only the semblance of existence, which, seen from this pure region bathed in the beams of the spiritual sun, merely reflects back its shades of color, its varied tints and lights, softened away into eternal rest" (*Philosophy of Religion,* English translation, Vol. I, p. 3). So Hegel. Bradley says: "We can see at once that there is nothing more real than what comes in religion. The man who demands a reality more solid than that of the religious consciousness, knows not what he seeks" (*Appearance and Reality,* p. 449). So when we talk of intuitional truths we are not getting into any void beyond experience. Intuitional experience is within the reach of all provided they strain themselves to it. These intuitional truths are not to be put down for chimeras simply because it is said that intellect is not adequate to grasp them. The whole, the Absolute, which is the highest concrete, is so rich that its wealth of content refuses to be forced into the fixed forms of the intellect. The life of the spirit is so overflowing that it bursts all barriers. It is vastly richer than human thought can compass. It breaks through every conceptual form and makes all intellectual determination impossible. The real is no more a pulseless identity excluding all differences; nor is it a chaotic disconnectedness with no order in it. It is the spiritual life, embracing the facts of nature which are shot through and through with the forms of mind. Philosophy is neither purely conceptualist nor merely empiricist but is intuitional. Art is the living expression of the soul which feels

itself to be in tune with the infinite. Morality is no more self-satisfaction or blind obedience to a set of categorical imperatives but is the life of a soul which feels its grip firmly on the spiritual destiny of the world. Philosophy, art and religion become different expressions of the one feeling of unity with the universe. This feeling of the essential oneness of the world-spirit failed the facts in the lower stages and made them lower, but now the identity is revealed and the Absolute is reached.

The relation of this Absolute *Ananda* to the other categories is one of higher to lower. The lower is included in the higher. The whole world is in *Ananda,* "The other beings live upon a small part of this *Ananda.*" This joy is the reality or essence of the lower categories. "Life is the essence of food, mind of life, knowledge of mind, joy of knowledge" (*Maitryana Brahmana Upanishad,* VI, *Prapathaka,* 13). The highest and the most concrete category is *Ananda.* All the rest are imperfect revelations of it.[4] The whole variety of being rests in the Absolute and "is an evolution from that alone" (*Bhagavat Gita,* XIII, 30). The Chandogya Upanishad says: "From the Self is life, from the Self is desire, from the Self is love, from the Self is *Akasa,* from the Self is light, from the Self are waters, from the Self is manifestation and disappearance, from the Self is food" (VII, 26). Ultimately, life, mechanism, consciousness and intellect are parts of this comprehensive whole. They are all abstracts from it and the Absolute is the only *res completa.* It is the only individual. We cannot attribute a substantial existence to the individuals of sense. If we do so we remain, to use Spinoza's language, at the level of imagination without rising to the level of

[4] The categories cannot adequately bring out the nature of Brahman though they all rest in it. "That which is not expressed by speech and by which speech is expressed;....that which does not think by mind and by which, they say, mind is thought;....that which does not breathe by breath, and by which breath is drawn, that alone know as Brahman, not that which people here adore" (*Kena Upanishad,* I, 5, 6 and 9).

reason. The Absolute therefore is the whole, the only individual and the sum of all perfection. The differences are reconciled in it and not obliterated. The dead mechanism of stones, the unconscious life of plants, the conscious life of animals and the self-conscious life of men are all parts of the Absolute and its expression at different stages. The same Absolute reveals itself in all these. The ultimate reality sleeps in the stone, breathes in the plants, feels in the animals and awakes to self-consciousness in man. It progressively manifests itself in and through these particulars. The Absolute thus is an organized whole, with interrelated parts. It embraces time, its events and processes. The finite universe is rooted in the Absolute. Life, mechanism, etc., are all members together of one whole. The Absolute is not an abstract unit but a concrete whole binding together the differences which are subordinate to it. The whole has existence through the parts, and the parts are intelligible only through the whole.

On this view there cannot be any "creation." The question as to why the Absolute limited itself, why God became man, why the perfect became imperfect, is irrelevant. For there is no such thing as an infinite which first was an infinite and then transformed itself into finite. The infinite is finite. The Absolute is the self and its other. Gaudapada in his Karikas on the Mandukya Upanishad mentions the different theories of the creation of the universe. The universe may be the creation of an extra-cosmic God, or an illusion or the product of evolution. He dismissed these theories as incorrect, and declared that it is of the nature of God to express himself. It is the essence of spirit to manifest itself. The world is the affirmation of the Absolute. The universe is the energizing of God. God realizes himself in the world. We do not have the infinite and the finite, God and the world, but only the in-

dynamically viewed. If eternity is a circle, then the process of the universe may be viewed as a straight line. The universe of finite objects gives us a moving image of eternity, in the words of Plato. The eternal is viewed as a growth or a becoming or a working out. In the universe we have the self-evolution of the Absolute. The lower stages, which are imperfect as compared with the higher, strive to become perfect. The whole universe is a vast struggle to realize the unity which is the ideal. This tension of the universe is mirrored in man, reflected in his individuality. The Taittiriya Upanishad declares that man is a microcosm in which all parts of reality are represented on a reduced scale.[8] His nature reaches up to the Absolute and down to the plant and the animal. While confined to a material organism, the individual self has the capacity to rise beyond intelligence into immediate contact with the divine. To bring about the unity between the higher and the lower is the aim of the individual self as it is the aim of the universe. The individual self is the theater in which is enacted the drama of the universe, namely, the realization of a central identity in and by means of the differences of mechanism and life, consciousness and intellect. The impulse toward union and harmony is present in all finite objects. The finite strives to pass out of itself. All objects of the universe are thus double-natured. "Whatever being is born, the unmoving or the moving, know thou, O best of the Bharatas, that to be owing to the union of *Kshetra* and *Kshetragna*, 'matter and spirit, finite and infinite'" (*Bhagavat Gita*, XIII, 26). They are finite-infinite. The finiteness *qua* finiteness is a standing contradiction to the infiniteness. The presence of the infinite enables the individual to break the finite and proceed higher up. It is by

[8] In Chapter II it is said that the individual should not be identified with either the physical or the vital or the mental or the intellectual self. The essence of the individual's nature is to be found in the self of bliss which is the inmost self of all.

such a breaking of the shell of finiteness that the infinite self finds itself and develops. To gain the higher we must give up the lower. Unless our little self is sacrificed, progress is not possible. Every step on the upward path of realization means sacrifice of something else. This sacrifice, which means friction, opposition and pain, is the penalty we have to undergo in rising to our selves, on account of our finiteness. Throughout we have these incidents in the growth of a soul. Pain and suffering are phases of all progress. The process of the life of self is also a process of death. To have the fruit we must sacrifice the flower, though it is hard and painful to sacrifice it. Evil is thus organically related to the higher interests of man and is a necessary phase in the development of the individual self. Evil is therefore as real as the finite being is real. In this universe there is always development. We can never say "it is finished." The Absolute is never in history completely revealed. If so there will be no universe and no finiteness. As Schelling says, "God never is, if *is* means exhibition in the objective world; if God were, we should not be." Again, "The ultimate goal of the finite ego and not only of it but of the non-ego—the final goal therefore of the world—is its annihilation as a world." As Bradley says, "Fully to realize the existence of the Absolute is for finite beings impossible. In order thus to know we should have to be and then we should not exist." When we see Brahma we become Brahman. That is the verdict of the Vedanta philosophy. As finite we cannot see; when we see, we become infinite. In the finite universe there will ever be approximation to the goal of reaching the infinite and never realization. The Absolute in this world is half dream, half reality. The universe is only a partial revelation of the Absolute. Knowledge is an infinite progress; morality, a ceaseless growth. That is why the Vedanta philosophy considers this finite world to be a be-

ginningless and endless *Samsara*. We can never completely break the shell of egoism and attain the infinite if we remain in the finite universe, giving a substantial existence to our own individual self. The release from this world of trouble, risk and adventure can be had only by losing the separate self. Absolute surrender of self to God, a perfect identification with the divine will, will "let us pent-up creatures through into eternity, our due." The Swetswatara Upanishad says: "In this wheel of Brahman, which is the support as well as the end of all beings, which is infinite, roams about the pilgrim soul when it fancies itself and the supreme ruler different. It obtains immortality when it is upheld by him" (i. e., when the soul thinks itself to be one with him" (V, 6). If the soul does not gain this height of spiritual splendor when it loses itself in the all, it will find itself again and again taking births in the finite universe, as a separate self with all the results of the past *Karma* entering into its nature. It will revolve in the wheel of births and deaths until it reaches the highest, when it gives up all subjection to time.

Pain and suffering then are necessary incidents in the development of a human soul, which, as given, is a discord. Man is at a parting of the ways. There is a conflict between the different elements, the higher and the lower. Man is the completion or fulfilment of the lower and the anticipation of the higher. But growth means the death of the lower and the birth of the higher self, and so it will be accompanied by the agony of death and the travail of birth. We have moral evil and sin if the finite self assumes a false sufficiency and independence and adopts a more or less indifferent, if not a hostile, attitude to the universe at large. He is a sinner who, owing to imperfect understanding, takes up a false defiant attitude to the not-self. Intellectually this act is error and morally it is evil. If a man considers his supreme good to be in the satisfac-

tion of his appetites and the desires of the organism, he is a sinner. Selfishness is the root cause of sin. It is the opposition of the finite to the infinite, the rebellion of man against God. Evil is as necessary as any other finite element in the universe. A universe without it will be a universe where the finite is swallowed up in the infinite. A mere infinite without finite is an impossible conception. Therefore evil is a permanent factor in the universe.

S. RADHAKRISHNAN.

MADRAS, INDIA.

THE CONCEPTION OF BRAHMA.

THE PHILOSOPHY OF MYSTICISM.

> **"In the whole world there is no study so beneficial and so elevating as that of the Upanishads. It has been the solace of my life, it will be the solace of my death."**
>
> **—Schopenhauer.**

THE Vedantic system of philosophy has two broad aspects, the esoteric and the exoteric. The former is technically metaphysical and is abstract in form; the latter is in a concrete historical setting and is for the requirements of those who have not, so to speak, risen above faith and form. The four main divisions of Vedantism deal with the doctrine of God or of the philosophical principle, the doctrine of the world, the doctrine of the soul, and lastly the doctrine of the fate of the soul after death. These constitute respectively the theology, cosmology, psychology and eschatology of the system. A treatment of these doctrines as such must proceed entirely on historical lines so as to represent faithfully the traditional views. But with this merely exoteric aspect we shall not be concerned at all, the present exposition being confined solely to the Vedantic theory of being, the central ontological tenet of the identity of the self and the universe, the doctrine of Brahma as the one and sole ultimate reality, the One Eternal Being to which there is no second.

In fact the attempt will be made to show how it is

possible to lead up, on the speculative side, to the great metaphysical truth of the unity of the cosmical principle of the universe and the self, a truth first grasped only intuitively by the mystics. The whole of Eastern mysticism, or for that matter of any mysticism, may be summed up in the compound word *Brahma-atma-aikyam,* i. e., the unity of the Brahma and the self. The significance of this is that there is only One real being, a Being that is absolutely One, and as the Vendantist goes on to add in his famous formula, *Tat tvam asi,* "That art Thou". The self or soul in each of us, this is the Absolute. But there is not a plurality of selves. There is only One, and That art Thou. Thus boldly the Hindu philosopher declares *Aham Brahma asmi,*—"I am Brahma." Thus does he identify the individual self with the eternal principle of all Being. Or, if one prefers to use the word God, there is naught but God and that art thou. The individual self is not a part of the Absolute nor an emanation from him, but it is absolutely identical with him.

And it is the Absolute here and now, though, owing to *Avidya* or ignorance, the illusion of plurality and separateness from the eternal indivisible Brahma results. Here in these few words we have presented to us the whole story of the Vedanta, which is endlessly repeated in ever varying forms throughout the Upanishads. But the theory as stated above is too condensed and requires fuller elaboration for any intelligible appreciation of it.

The fundamental conception of the identity of the self and the universe was arrived at intuitively rather than by metaphysical speculation. But let it not be supposed that because the Hindu sages reached this truth in the first instance mystically, therefore it cannot be defended on rational grounds or in fact even arrived at by way of reason; for philosophical mysticism is as much a rational theory as any speculative philosophical theory is, and can justify

itself in terms of discursive thought. The conclusion reached by the Vedantist that the process of ideation is essentially defective and must therefore be transcended, does not make the theory any the less philosophical or the arguments any less cogent. There is no weight in the objection that arguments showing the unsatisfactoriness of the thinking process must thereby be invalid. When it is declared that the individual self alone is, there is an obvious danger of the mystic position being confused with mere solipsism. According to the solipsist, what appear to be other finite selves like himself are in reality merely his experience. There are no other selves, only he exists. Now the Vedantist in affirming the sole reality of the *Atman* does not say that other selves are merely his experience and that there is naught beyond his present self and its experience. What he does is to identify himself with other selves, and even further with all else. The doctrine here seems to be merely realistic, for though the view taken of being is monistic, yet the Absolute does not differ from the realistic One of the Eleatics. Both the reality and the observer of it are regarded as real.

But at this point through the very realistic form we see the transformation that has been effected, for the world is here identified with the observer and with him in so far as he is the knower of the unity. There is then no external world independent of knowledge, for it is the very knower in so far as he knows, and thus what was apparently a merely realistic monistic doctrine is seen to be really not so, becoming completely idealistic at a stroke in the identification of the knower and the universe. The illusion for the solipsist is the other selves and whatever else he considers not himself. For the Vedantist, on the contrary, all this is not illusory; the illusion consists in his thinking that they are other than himself. It is the illusion of separateness, of diversity. For the solipsist the things are illusory; for

the Vedantist not the things but the plurality is illusory. Thus there is a world of difference between the two positions, though both agree in declaring the sole reality of the self.

Being is defined as an absolute and simple unity by the Vedantist. The manifoldness is merely illusory, or a "mere matter of words" as the Upanishads express it. Therefore is the Absolute distinctionless, without attributes, unconditioned, and since knowledge involves the duality of subject and object and the Absolute forms a unity, it is also unknowable. Knowledge must be transcended to obtain oneness with the Brahma or *Atman*. So in speaking of the Absolute, which is the self, the Hindu says, "Before him words and thought recoil not, finding him." All that can be said of him is *Neti, neti* ("It is not so, it is not so").

What then is the nature of this reality? Since by reason of our intellectual constitution we cannot know it, how then can Brahma, the eternal and indivisible, be apprehended? The answer is given in the following stanzas of the Kathaka Upanishad:

> "Not by speech, not by thought,
> Not by sight is he comprehended;
> 'He is,' by this word is he comprehended
> And in no other way.
>
> "'He is,' thus may he be apprehended
> In so far as he is the essence of both.
> 'He is.' To the man who thus has apprehended him
> His essential nature becomes manifest."

Thus we see that to be real means to be immediate so completely that knower and known, subject and object become one, so that all thought and ideas, being absolutely satisfied, are transcended. Since there is no sundering between knower and known, here knowing and being are one. It is the unique immediacy of the awareness of the inner self. "I am I" is all that can be said. The knowledge is not mere descriptive knowledge, for even if I were

to be familiar with all that science could ever teach, I would be no nearer to my inner self, the gulf would not be bridged. But furthermore it is not even knowledge by acquaintance, that knowledge by which we are directly aware without the intermediary of any process of inference or any knowledge of truths, in other words awareness of sense data, of brute facts; for here too there is as complete a sundering as in the case of knowledge by description. The apprehension is of that unique nature whereby I can only say "I am I." However much knowledge by description or by acquaintance you may have of this table, it still baffles you for it is other than you; but you are aware in a totally different way of yourself because you are yourself. But the Brahma or self is unknowable in the ordinary sense of the word knowledge, i. e., intellectual relational knowledge, for all knowledge involves the duality of knowing subject and the object known, whereas the inner self can never be the object known. For in any act of cognition it is the knower. And this leads the Vedantist to say of it that all words and thought recoil, not finding it. This self cannot be proved, for in proving it you already presuppose it; nor can it be disproved, for according to the old Cartesian formula *Cogito ergo sum*, in the very denial of it you affirm its reality. Thus the self is absolutely inaccessible to our intellect, which belongs to this relational world. It is beyond any act of cognition, for in it subject and object are identified. It is inexpressible in terms of idea. To know it is to be it, for it cannot be the object of any finite thought. Now since Brahma is beyond all ideas he cannot be conceived as having any attributes. He is free from all determination. Nothing can be predicated of him for he is beyond the reach of finite thought, which, searching ever for an Other, implies the dualism of subject and object. In the self or Absolute, knower and known and knowledge are all one. All the opposition and contradiction of this

appearance-world is transcended in an absolute immediacy. And as Royce in his interpretation of the position says, "We must regard the absolute immediacy not as the raw material of meaning but as the restful goal of all meaning, —as beyond ideas, even because it is simpler than they are. It is at once nothing independent of knowledge and nothing that admits of diversity within knowledge. The self is precisely the very knower, not as a thing that first is real and then knows, but as the very act of seeing, hearing, thinking, in so far as the mediating presence of some Other, of some object that is known, seen, heard, thought, is simply removed, and in so far as the diversity of the acts of knowing, seeing, hearing, thinking is also removed.[1]

In attempting to trace out some definite line or argumentation by which the Vedantic conclusion of the sole reality of the *Atman* or self may be established, proof positive must not be looked for, for where any constructive effort is concerned it is only possible to open up lines of thought, to hint and to suggest rather than to establish propositions in any finally demonstrative manner. The hints and suggestions themselves are guided by the conclusion which has really already been arrived at intuitively.

Anything beyond a superficial investigation of the position reveals to us that Vedantism finds its bed-rock in a criterion of reality which is not only universal but also absolutely certain, for self-contradition results from doubting it. The principle when brought to light is that reality is self-consistent and internally coherent, that it does not and cannot involve self-contradiction. But reality does not here and now present itself to us as free from self-contradiction. Royce has in his Gifford lectures ably described "the finite situation that sends us all alike looking for true being." In this situation in which we finite beings find ourselves there is ever a conflict between mere immediate brute facts

[1] *The World and the Individual*, Vol. I.

or meaningless experience and idealized experience or that possessing meaning. We are ever confronted with the contrast of fact and idea, and the world-process consists in trying to win one side over to the other, to illuminate blind brute facts with the light of meaning; in other words to realize more and more that reality is not self-contradictory. Briefly we may look upon the conflict as the effort of thought to comprehend being, the attempt at a reconciliation between knowledge and being.

In this disquieting situation we seek for an Other, which if found would end the conflict, and in the winning of which the meaningless would vanish and thought have accomplished its task. We seek to make our ideas complete embodiments of meaning instead of leaving them in their present state of partial embodiment. We have in our finite situation merely relative immediacy, for both masses of sensation and feeling, which are the meaningless aspect of our ordinary consciousness, and ideas, which are relatively meaningful, are not wholly immediate because they are not wholly satisfying. The intellect and will are not to be sundered in an abstract fashion, for, as has been truly said, all our conscious deeds are merely immediately visible and tangible ideas, and thoughts are nascent deeds. Thus in this disquieting situation of merely partial immediacy and satisfaction, we search restlessly for an Other to end the quest, for some final and wholly satisfying fulfilment. What we seek is something to end our disquietude, for till this Other, which we finite beings just because of our finitude ever pursue, is won, reality must remain largely incoherent and meaningless. That which is real therefore must not, when confronted, involve the finite striving of thought and desire, for these by their very constitution and by their presence imply the admission that truth is not present in its totality. But, it will be objected, why should the self-consistent nature of reality be present

to us? Is it not sufficient that it should be self-consistent and yet beyond us altogether, completely out of our reach? This is the course adopted by the realist, and to him we must now turn our attention.

Realism is fully aware of the above finite situation which brings dissatisfaction and forces us to admit that truth is not present to us. But taking the very opposite direction from Vedantism it makes reality an independent being absolutely beyond all our striving, for it is defined as independent of all knowledge that refers to it. By realism we may understand any theory that sunders the object from the idea of that object, that is, which asserts that reality is not dependent for its existence upon the ideas or states of consciousness of the knowing subject. More precisely realism is the doctrine that makes the essential character of real objects to be their independence of all actual or possible external knowing processes whatever. Independence of knowledge that refers to it from without, this is the mark of a real object. It is evident from this that reality need not be matter, for realism can just as well be immaterialistic, as in the case of the monads of Leibniz, the things-in-themselves of Kant and the Platonic ideas. There is no need to discuss any special forms of realism, for since the argument is directed against the very ontological predicate itself and not to the objects to which it applies, it little matters to us whether the real beings are conscious monads or atoms or material substances. The attack is against a world of independent beings, of whatever type these beings may be. Realism asserts that our knowledge of a thing makes no difference to it. It is the object which can make our ideas of it true or false, but the truth or the falsity of our ideas does not affect the object itself. Real being is supposed to be independent of knowing and yet capable of being known. But epistemological considerations show quite clearly that the object

cannot thus be sundered from the idea of it. Reality is known to me only through my intellect,—the world is presented to me as experience or psychical matter of fact. Knowledge and being are for us co-extensive. Whatever it be in itself, for me at all events the world is my representation, for apart from the forms of my intellect which it presupposes, it has no reality. Make the attempt to think of anything whatever as real and yet outside of all experience and the absolute futility of trying to sunder knowledge and reality will be realized. The very reality of a thing consists in its being known, for we cannot get outside our own experience. As a matter of fact both the real and the unreal are defined in the same way by the realist, for according to him reality is independent of any knowing, and it will be found that the unreal cannot be thought of otherwise than as that of which no mind is ever aware. Thus if the real and the unreal are not to be considered identical, then reality cannot be independent of experience. Idea and object must not be severed, and the dualism of the realistic view has to be abandoned. The principle of the inner consistency of reality cannot be realized by means of it. An Other entirely beyond us cannot end the disquietude of the situation in which we as finite beings find ourselves. The contrast of fact and idea must be overcome in some other way, for if reality is in very truth not self-contradictory the realistic explanation fails to satisfy. Here steps in the Vedantic mystic saying that the disquietude and contradiction of relational thought cannot cease as long as there is an Other involved. He is not content with half measures. If there is any sundering or separateness, if there is a vestige of otherness remaining, we do not get nearer than mere knowledge by acquaintance. There will still be dissatisfaction, actual or possible,—at least the possible dissatisfaction of not being able to occupy the standpoint of that which is other than you. The ob-

jection could not be brought forward that you could occupy the standpoint of that which is other than you, for if you did the Otherness would disappear and leave only the self, and this means coming round to the Vedantic position of the unity of the self and the universe. Thus in the realization of the Absolute, if there is to be an end to the disquietude arising from finitude, it can only be in some sort of ineffable immediacy in which all otherness disappears, in which very thought and reason are quenched. So the Vedantist speaks of the *Atman* or self as its own light, "the light of lights," even as Kant spoke of "the good will," the jewel that shines by its own light. In a superb verse in the Upanishads (I use Deussen's translation), the thought is expressed that

> "There no sun shines, no moon, nor glimmering star,
> Nor yonder lightning; the fire of earth is quenched;
> From him, who alone shines, all else borrows its brightness,
> The whole world bursts into splendor at his shining."

Thus the Vedantist comes to deny the manifold realities of the finite world. He says they are illusory. And why are they illusory? It is precisely because they cannot be independent of the knowledge of them, and this means that reality must be one; but since there must be no duality even in this One, therefore it must be knower and known and knowledge in One. Reality is not a sum of parts, not an aggregate of many, but all as one. The realist can also say that being is one. But the so-called monistic realist is really a dualist, for he still interprets the One as being independent of all knowledge of it. How would it be possible then to escape the pitfall which besets the realist? This is done by saying the world is one because its oneness is my oneness and I myself am Brahma, the world principle. I am the All. And I, as Brahma, am not independent of the idea that knows me for I am identical with it. Thus the absolute unity is at once absolute reality and absolute

knowledge. But this absolute knowledge excludes the dualism of subject and object, knower and known, and excludes every kind of synthesis and relation. The unity is not to be sought for without, for all search for an Other as is involved in finite thinking brings disquietude and contradiction. The unity is my unity and is therefore within. As Uddalaka, instructing his son and disciple, says so often, "Believe me, O gentle youth, what that hidden thing is, of whose essence is all the world, that is the reality, that is the soul, that art thou." But the way in which this self is, cannot be expressed in terms of our empirical knowledge. In winning oneness with it the very reason is quenched in an absolute immediacy, which is the cessation of all finite process of striving and thinking. The plurality involved in thought and desire is itself illusory. If then in very truth there is no variety, why does the Vedantist still behave as if there were diversity and manifoldness? The answer is that he himself, like any other being caught in the net of illusions, is struggling with them; and to him it is as if there were diversity, whereas really, if he could attain the higher transcendent standpoint, he would realize that there is none. But if it is asked whence comes this *Avidya* or ignorance through which we get entangled in *Maya* or the great world-illusion, the only answer is that the question is inadmissable, for the category of causality does not apply to what is beyond this world of our relational empirical knowledge. Causality itself is a part of the illusion. Now, as already stated, the One of the Vedanta cannot be reached by discursive thought, by means of our intellectual knowledge; the duality of subject and object must be transcended. This is why Brahma can be characterized only negatively. *Neti, neti*, it is not thus, it is not thus—is all that can be said from this lower standpoint, from this world of unrealities, this world of contradictions and oppositions. And since the Absolute is defined as ab-

sence of finitude, since all finite ideas about it are abandoned as vain, it is said that the Absolute is really equivalent to nothing. It is argued that the Absolute of the Vedantic mystic gets its very perfection from a contrast effect.

Mysticism as a conception of being is said to be a conscious abstraction and to be the logically precise and symmetrical counterpart of realism in that each doctrine seeks an absolute finality—a limit which is conceived solely by virtue of its contrast with the process whereby our ideas tend toward that limit, and that neither can tell what it means by its goal. Now with realism we have already dealt, but the criticism fails when directed against the Vedantic position. It is not justifiable to ask of the Vedantist what he means by his goal, for enough emphasis has already been laid on the Absolute inaccessibility of the Brahma to all empirical knowledge. For the Vedantist to know is to be, and therefore to tell the meaning of the goal would be to be the goal itself. Vedantism cannot escape from its finitude by words, ideas, by any intellectual relational knowledge, for these are finite. The defect is in them. Finite thought can lead you to posit a higher transcendent standpoint in which the sundering of subject and object is not involved. It can point beyond itself to an "ultrarelational intuition" by which the absolute unity may be grasped. But one must not expect to arrive at the Absolute by means of finite thought itself. Therefore it is inadmissible to demand of the Vedantist that he should define the content of his Real Being, for this simply means asking him to translate in terms of ideas what he has already said beyond all ideational process. All he can do is again to repeat that to know is literally to be and that therefore so far as empirical knowledge is concerned the Absolute can only be defined negatively.

In this connection one recalls the beautiful story in the Upanishads, where King Vashkali asks Bahva, the sage,

to explain the nature of Brahma to him. Thrice the king addressed him: "Teach me, most reverend sir, the nature of Brahma." But Bahva the Wise remained silent. And finally, when the king repeated his demand, he replied, "I tell it you, but you do not understand it; this *Atman* is silence." Thus Bahva sought to show that Brahma is not won by looking outward. Bound Prometheus-like to the frame-work of the categories and the innate forms of perception, we are shut out from an intellectual knowledge of Brahma, that which rises above all categories and forms of perception. But we come to God by absorption into our own self, for as so often repeated throughout the Upanishads, the Brahma is the self and I am Brahma,—a fearless synthesis indeed; but the seeker after truth does not dread the consequence of his search, for he "dares to be wise."

LEO C. ROBERTSON.

BURMAH, INDIA.

THE TRINITY.

"Die Dreifaltigkeitslehre vertieft den Begriff Gottes und macht dessen Vermenschlichung unmöglich.—*Ein deutscher Mystiker.*

STRANGE world, bewildering in its complex beauty
And yet so simple in its constitution!
Unfathomed in its depth and unexhausted
In possibilities of startling changes,
The universe remains an unsolved problem.
How varied in its forms, how infinite
In its unending whirls, original
In every spot and new at every moment,
Yet always all its laws remain the same!
And this unaltered, this unbroken sameness
Is rigid uniformity evincing
The simplest rules of truths self-evident,
Of axioms that are plain—as straight and clear
As are the rays which from the distant stars
Reach us like greetings from the worlds beyond,
Revealing to us by inspiring visions
The depth and grandeur of the universe.

Yea, straightness is the mystery of being;
The plainest, simplest facts present the problem.
Of all the riddles that confront the search
Of our unsatedly inquiring souls.
It is the simplest truth which baffles most.

Nature surrounds us. Like an open book
It lies before us, and we can decipher
Its most amazing and most intricate
Phenomena if we but understand
The simplest truths of its most certain laws,
Of laws that all are ultimately one.
In their innumerable applications
These laws produce varieties untold;
Yet they agree, they harmonize, and all
Remain one and the same in their unbroken
And their unaltered uniformity.

This uniformity throughout existence,
This omnipresent and intrinsic order
Patently simple and yet so profound,
Renders the world a wondrous cosmic whole,
And thereby makes the universe divine.
For its intrinsic oneness, systematic
And all-consistent—this is God. Aye this
And this alone, is God, the real God.

God is immutable and omnipresent.
He is the law supreme that never changes.
In truth, He is Eternity itself.

But God is more; God is not stagnancy,
Not tedious sameness nor monotony.
God is life's law, life's governor, life's guide,
He is the law in its eternal action.
God is the truth applied; He manifesteth
His very being as the world's creator.

Creation is the living God; creation
Proves God's existence; it is God at work;
In Nature God appears, and Nature truly

Is He himself. In Nature He reveals
And manifests His will. The universes,
Unfolding evolutionary life,
Are God made visible, God in the making.
'Tis God who stirs in genesis of being;
He is its actuality, and He
The law that dominates and molds its life,
The norm of Nature swaying its commotions.

'Tis God who comes to life as helpless babe
Ayearn for consciousness. 'Tis God who grows
In childhood and in youth. 'Tis He who struggles
In us for truth and righteousness. 'Tis God
Who is betrayed and bears the curse of sin,
Who suffers on the cross and meets defeat
In ignominious death, but from the tomb
He rises to triumphant victory.

So God is both Creator and Creation;
He is the Father and He is the Son,
He is Eternity and He is Time.
He is the Will immutable, yet also
Is He the stir of life, its constant change.

So God would seem to contradict himself,
To be at rest and yet to be in motion.
But no, the contrast in his being is
A higher unit, not a dualism.
There is no split in God's divinity.
The two are one, united in a Third.
This third is the eternal aim of God.
It is His purpose to be carried out;
It is the future of great things to be;
The spirit 'tis which animates ideals,

The plan it is of God's creative power,
The plan and the direction of His will.

What is the pulse that beats in human hearts?
What is the standard of our aspirations?
And what the guiding star that leads us onward?
The aim and hope that stablisheth our faith?
Is not this also God? It is God's spirit
That shines above as star of Bethlehem
To lead the Magi on the way to truth,
To newer truths of broader comprehension.
It is the longing for a higher life
That thrills the breath of martyrs. It is God
Who animates the world with sacred aims,
Inspires the hero to courageous deeds
And fills his anxious heart with confidence,
With noble purpose of self-sacrifice,
And gives him strength to die for his ideals.

Here lies the secret of that mystery,
That triune mystery, life's meaning, course and aim;
It is the trinity of cosmic order,
The trinity of God as Law supreme,
As God revealed in glorious self-creation
And as the aim and purpose of His work,
As the ideal to be manifested.
God, thou art One, but not one rigid unit;
Thou livest in the contrasts of existence,
And by whatever name we greet the last
And ultimate foundation of our being
We are but an effulgence of Thyself.

The God-intoxicated prophet claims
That "Thou art One, one only, unbegotten
And no begetter; Thou art God, not Father

And not a Son. Lord art Thou, Lord alone."
God, Lord and King, all-merciful, almighty,
Reveal Thyself, explain this deepest riddle,
The problem of creative deity!

And in my heart the Still Small Voice was heard;
It spake and answered, saying: God is God,
God in Himself alone would be complete,
But God, alone, would be mere non-existence;
He'd be a law that finds no application,
The All and Naught unlimited and blank,
The infinite and zero all in one.
God to be God, to be an actual God,
Must manifest Himself, must live and work,
For He appears alive but in creation.
Thus only God becomes concrete in form,
Thus only He reveals His dispensation.

The wild commotions of a gaseous whirl
Change slowly into planetary systems,
As all the turbulent and glowing masses
Obey mechanic laws of cosmic order.
Yea laws mechanic, necessary laws,
Those truths eternal, are the thoughts of God;
Eternal thoughts, thoughts of the Overgod.

God moveth step by step according to
Th' eternal norms which constitute His being;
And on the paths prescribed by God Himself
Creation struggles higher, ever higher,
To life and consciousness with joy and pain.

O God, Thou art not merely fashioner
Of clocklike universes, nor art Thou
An ego unit like a mortal man,

then any term in the geometrical progression is greater than the corresponding term in the arithmetical progression. Hence Barrow concludes that if, in the theorem above, B is not greater than M, then F must be less than Q, which is contrary to the hypothesis. He then deduces that, if F = Q, then B > M, C > N, and so on.

Thus Barrow, and no more; now let us see what he might have got out of this if he had so chosen.

If Barrow's final conclusion is expressed differently we have:

Fig. 1.

Suppose that a straight line AB is divided into two parts at C, and the part CB is divided at D, E, F, G in Fig. 1 (i), and at D', E', F', G' in Fig. 1 (ii), so that AC, AD, AE, AF, AG, AB are in arithmetical progression, and AC, AD', AE', AF' AG', AB are in geometrical progression; then AD > AD', ... AG > AG'.

Expressing this algebraically, we see that, if AC = a, and CB = $a.x$, and the number of points between C and B is $n-1$, and H is the rth arithmetical and H' is the rth geometrical "mean" point; then the relation AH > AH' becomes

$$a+r.ax/n > a.[\sqrt[n]{\{(a+ax)/a\}}]^r;$$

$$\text{i. e., } 1+x.r/n > (1+x)^{r/n}; \text{ where } n > r.$$

Also, as CB becomes smaller and smaller the inequality tends to become an equality.

Moreover, if we put $rx/n = y$, and hence $x = ny/r$, then

$$1+y.n/r < (1+y)^{n/r}; \text{ where } n > r;$$

and the inequality tends to become an equality.

Naturally a man who uses the notation xx for x^2 does not state such a theorem about fractional indices. But none the less he has the approximation to the binomial theorem; that is, all that is necessary for him to obtain the gradient of $x^{n/r}$ or $x^{r/n}$, where $n > r$, although it is concealed in a geometrical form. We may as well say that the ancient geometers did not know the expansion for sin(A+B), when they used it in the form of Ptolemy's Theorem, as say that Barrow was unaware of the inner meaning of his proposition. Also from the *a fortiori* method of his proof it is evident that he knew that the relative error was less than x/n. It may be objected that

this is insufficient to make the relative error negligible, no matter how small x may be. But these old geometers could use their geometrical facts with far greater skill than many mathematicians of to-day can use their analysis. Barrow does not require to know the magnitude of the error at all; he only requires to know that the inequality in the above example is always in one direction, i. e., the geometric always less than, or always greater than, the corresponding arithmetic mean. The way in which the theorem is used, which indeed is his general method for drawing tangents, is of striking ingenuity. Barrow starts with a very small, so to speak, stock-in-trade; he is able to draw a tangent to a circle, and also to a hyperbola of which the asymptotes are known, and he has the fact that a straight line is everywhere its own tangent. The tool that he most often uses is the hyperbola; and when he cannot immediately find a construction for a tangent to a curve, he draws a hyperbola to touch the curve, and then draws the tangent to the hyperbola. His criterion of tangency is the following:

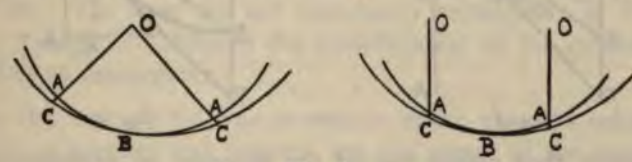

Fig. 2.

A straight line and a curve, or two curves, will touch one another if one curve lies totally outside or inside the other line. That is, the curves ABA, CBC, touch one another, if OA < OC, whether O is supposed to be some fixed point, or the straight lines CAO are all drawn parallel to some straight line fixed in position. This criterion is important, as it will be referred to later.

In the next example chosen he does not however use any of the above three tools; for, finding that the curves formed from the arithmetical and geometrical means of the same order are such that he can draw a tangent at any point of the former in a very simple manner, he uses this as his auxiliary curve to find the tangent at any point of the latter.

Example 2 (*Lectio* IX, 1).

"Let the straight lines AB, VD be parallel to one another; and let a straight line DB, given in position, cut them; also let the lines

EBE, FBF pass through B and be so related that, if any straight line PG is drawn parallel to DB, then PF is always an arithmetical mean of the same given order between PG and PE; also let BS touch the curve EBE. It is required to find the tangent at B to the curve FBF."

The construction given is:

Make DS:DT = FG:EG; and join BT. Then BT is the required tangent (see Fig. 3).

The proof is as follows:

FG:EG = DS:DT = LG:KG; hence, since KG < EG, ∴ LG < FG. Therefore BT is the tangent.*

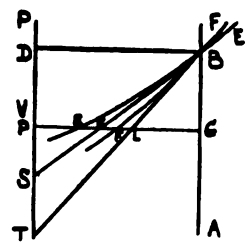

Fig. 3.

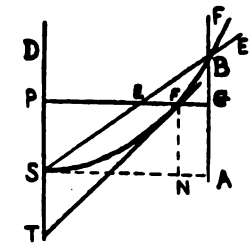

Fig. 4.

Barrow then makes use of the theorem on arithmetical and geometrical means, given as our first example, to show that the same construction holds good if PF is a geometrical mean of the same order between PG and PE, by proving that the curve formed from the geometrical means touches the curve formed from the arithmetical means at B. Lastly, he shows, by the use of an analogous curve, that a similar construction can be used for drawing the tangent at any point F on the curve FBF, provided that the tangent at the corresponding point E on the curve EBE is known (see Fig. 4). He then adds the remarkable note:

"It is to be noted that if EBE is supposed to be a straight line, the line FBF is one of the parabolas or paraboliform curves. Wherefore, what is generally known about these curves (deduced by calculation, and verified by a sort of induction, yet not anywhere proved geometrically, as far as I am aware) flows from an im-*

* This undoubtedly refers to the work of Wallis.

* Note, in passing, that this is equivalent to saying that the gradient of $f[x.r/n + a.(n-r)/n]$ is r/n times the gradient of $f(x)$ at the point where $x = a$.

mensely more fruitful source, and covers innumerable curves of other kinds."†

Now if, in Fig. 4, which shows Barrow's method of drawing the tangent at any point F of the paraboliform FBF, we take SA and SD as the axes of coordinates, and suppose that PF is the rth mean, out of n means,‡ between PG and PE, so that $PT : PS = n : r$, and $SA = a$, $PE = b$, $SP = mb$, where m is the gradient of EBE; then for the curve FBF, we have

$$y = FN = SP = mb; \text{ and } x = SN = PF = a.(b/a)^{r/n} = b^{r/n}.a^{(n-r)/n};$$

and the equation to the curve FBF is

$$(y/m)^{r/n} = x/a^{(n-r)/n} \text{ or } y = K x^{n/r};$$

whilst the gradient of the tangent at F is

$$PT/PF = (n/r).(PS/PF) = (n/r).(y/x) = (n/r).K x^{n/r-1}.$$

Thus the gradient is found for any curve of the form $y = K x^{p/q}$, where $p > q$; and, by interchanging the axes, for any curve of the form $y = K x^{p/q}$, where $p < q$.

Note. The axes are not necessarily rectangular in Barrow's figure; though of course in the consideration of the gradient they are taken as rectangular.

In the face of the note quoted in italics above, I submit that it is idle to contend that Barrow was not aware of the significance of his theorem; but as before, he was not prepared to use the index notation, let alone fractional indices. For this reason, most probably, he also leaves the point that, if EBE is a hyperbola, so that PS.PE is a constant, m say, then $y = m/b$, and the equation of the curve FBF is of the form $y = K x^{-p/q}$.§

Thus Barrow proves geometrically and rigidly, without any difficulty about the convergence of the binomial theorem, that in general, if $y = K x^n$, then $dy/dx = n.y/x$. He could have drawn the tangent, or found its gradient, by the method which he either thought little of, or affected to despise—*ex calculo* (observe the half-sneering comparison between the methods of calculation adopted by Wallis (?) and a geometrical proof, in the parenthesis in Barrow's

† In other words, the gradient of $f(x^{r/n}.a^{(n-r)/n})$ is r/n times the gradient of $f(x)$, at the point where $x = a$.

‡ It should be observed that Barrow defines previously such a curve as the locus of F as "having an exponent r/n."

§ He does this in a considerably harder way in *Lectio* IX, 10; from this general theorem the case when EBE is a straight line is deduced in exactly the same way as for the paraboliforms, and yields the hyperboliforms $y = K x^{-p/q}$.

note, as quoted above). Thus Barrow is in possession of a method for differentiating any explicit algebraic function of x; for he has another theorem connecting the tangents to two allied curves, the ordinate of one being proportional to a power of that of the other. For instance, he could have differentiated such a function as

$$(x+a)^{2/3}+(x^2-a^2)^{3/4}.$$

Of course Barrow does not consider such a case as this; at least, he has not got a theorem to draw a tangent to a curve, whose ordinates are the sum of the ordinates of two other curves, of which the tangents at every point are known.* Such a construction is easy; but the point I make is that Barrow was in a position to do any differentiation of this kind, by calculation, if he had had a mind to.

Further, by combining this method with the "differential triangle" method (the well-known "a and e" method—the prototype of the "h and k" method of the ordinary beginner's text-book of to-day), he could have differentiated implicit functions also, again by calculation. As examples of the "differential triangle" method Barrow takes the *Folium of Descartes* and the *Quadratrix* amongst others. A third example is of even more interest. Barrow finds the subtangent of a curve, which turns out to have an equation $y=\tan x$; moreover, he leaves it in such a form (namely, $t:m=rr:rr+mm$), that it is only necessary to put $r=1$ and $m=y$, in order to obtain

$$dy/dx=m/t=1+y^2=1+\tan^2 x=\sec^2 x.$$

In addition, the pair of figures that he gives could equally well have been used to find the subtangent for $y=\sin x$, in a form that immediately yields $dy/dx=m/t=\cos x$; but he winds up by saying, "These would seem to be sufficient to explain this method."

It is of course well known that Barrow was the first to perceive that differentiation and integration were inverse operations. This is proved in a very simple manner by means of a theorem and its converse.

In Fig. 5, ZGEG is a curve such that the ordinates to an axis VD continually increase (or decrease) from left to right. VIFI is

* This ability to deal with irrational algebraic functions, and that too without the binomial theorem, constitutes perhaps Barrow's greatest advance on the work of his predecessors on the infinitesimal calculus; although it by no means constitutes his only claim to great genius.

another curve, constructed from the former in such a way that the rectangle contained by the ordinate DF and a given length R is always equal to the area intercepted between the ordinates VZ and DE.

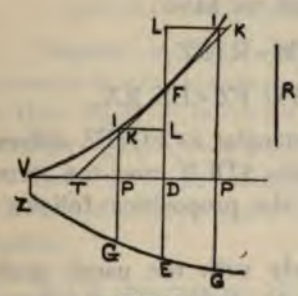

Fig. 5.

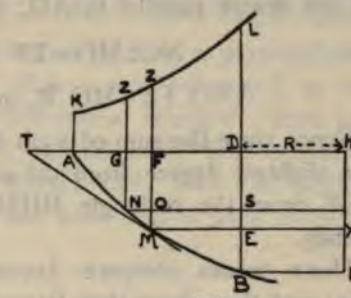

Fig. 6.

Then, completing the figure as above, and making DT : R = DF : DE, we have LF : LK = DF : DT = DE : R (by construction):

$$\therefore \text{LF} \cdot \text{R} = \text{LK} \cdot \text{DE};$$

but, by hypothesis, LF . R = area PDEG

$$\lesseqgtr \text{DP} \cdot \text{DE} \text{ (as P is on } \begin{matrix}\text{left}\\ \text{right}\end{matrix} \text{ of D)}$$

$$\therefore \text{LK} \lesseqgtr \text{DP, i. e., } \lesseqgtr \text{LI (“ “ “ “ “ “ “)}$$

and therefore KFK touches VIFI at F.

Cor. It is to be observed that DE . DT = space VDEZ.

Now if we call the general ordinate of the curve VGEG, y, and the general ordinate of the curve VIFI, y_1, this theorem becomes:

If by construction we are given that

$$\int y\,dx = \text{area VDEZ} = \text{R} \cdot \text{DF} = \text{R} \cdot y_1;$$

then $dy_1/dx = \text{FL}/\text{LK} = (\text{area PGED}/\text{R})/\text{LK} = \text{DE}/\text{R}$,

$$\text{i. e., } \text{R} \cdot dy_1/dx = y.$$

The converse theorem is thus stated and proved:

Let AMB be a curve of which the axis is AD, and let BD be perpendicular to AD (see Fig. 6). Also let KZL be another curve such that, when any point M is taken in the curve AB, and through it are drawn MT, a tangent to the curve AB, and MFZ, a parallel to DB (cutting the curve KL in Z and AD in F)—and R is a line of given length—then TF : FM = R : FZ always. With these data,

the space ADLK shall always be equal to the rectangle contained by R and DB.

For if DE = R, and the rectangle BDHI is completed, and MN is taken to be an indefinitely small arc of the curve AB, and MEX, NOS are drawn parallel to AD; then we have

$$NO : MO = TF : FM = R : FZ;$$

$$\therefore NO.FZ = MO.R, \text{ or } FG.FZ = ES.EX.$$

Hence since the sum of such rectangles as FG.FZ differs only in the slightest degree from the space ADLK, and the rectangles ES.EX from the rectangle DHIB, the proposition follows quite obviously.

These proofs compare favorably with the usual analytical proofs; and they show that Barrow not only appreciated the fact that differentiation and integration are inverse operations, but also recognized the necessity of proving the fact both directly and conversely. As I have mentioned, this is fairly well known; but what does not seem to have been remarked is that Barrow ever made any use of the theorems. However in the appendix to *Lectio* XI, where he develops the work of Huygens on the measurement of the circle, Barrow quotes formulas for the area and the position of the center of gravity of any paraboliform; but he states *"of which the proofs follow without much difficulty in various ways from what has already been shown,"* and leaves the rest to the reader. As a matter of fact, the proofs do follow quite easily, as is shown below; moreover Barrow could have found the radius of gyration of a paraboliform, or other power summations, practically amounting to $\int y^n dx$, by means of theorems previously given.

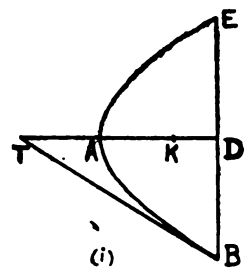

Fig. 7 (i)

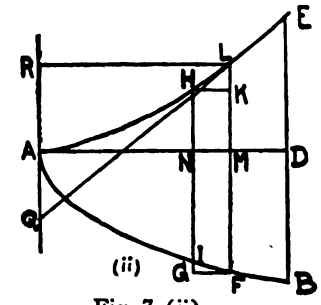

Fig. 7 (ii)

"If BAE is a paraboliform curve whose axis is AD and base or

ordinate BDE, BT a tangent to it, and K the center of gravity; then, if its exponent is n/m, we have

$$\text{Area of } BAE = m/(m+n) \text{ of } AD.BE;\ TD = m/n \text{ of } AD;$$

$$\text{and } KD = m/(n+2m) \text{ of } AD.\text{"}\quad [\text{See Fig. 7 (i).}]$$

Suppose, in Fig. 7 (ii), that AHLE is a paraboliform whose exponent is $r/s = 1/a$, say; let H be a near point to L on the curve, so that HLK is Barrow's "differential triangle"; then $LK/HK =$ gradient $= QR/RL = a.AR/RL = a.LM/AM$; and conversely.

Let AIFB be another curve, such that $FM/R = LK/HK = a.LM/AM$ always, then, as has been shown, area $AFBD = R.DE$ always.

But in this case we have

$$IG : FM = LM/AM - HN/AN : LM/AM,$$

$$= AM.LK - LM.HK : LM.AN,$$

$$= (a-1).LM.HK : LM.AN;$$

$$\therefore FG/GI = 1/(a-1) \text{ of } AM/FM.$$

Hence AIFB is a paraboliform, vertex A, axis AD, and exponent equal to $a-1$. Conversely, if AIFB is a paraboliform whose exponent is $n/m\ (=a-1)$; then the integral curve AHLE is a paraboliform whose exponent is $1/a$ or $m/(n+m)$; and since $DB/R = a.DE/AD$, the area $AIFBD = R.DE = m/(n+m)$ of $AD.DB$.

Similarly, area $ALED = AD.DE - (n+m)/(n+2m)$ of $AD.DE$

$$= m/(n+2m) \text{ of } AD.DE;$$

$$\therefore R.a.\text{area } ALED : AD.\text{area } AFBD = n+m : n+2m.$$

Now since $FM/R = a.LM/AM$, $\therefore FM.AM.MN = R.a.LM.HK$; hence, summing, we have $AK.\text{area } AFBD = R.a.\text{area } ALED$;

$$AK : AD = n+m : n+2m, \text{ or } KD = n/(n+2m) \text{ of } AD.$$

In a similar manner the radius of gyration could have been found from the sum of $FM.MN.AM^2 = R.a.LM.HK.AM$; and so on for higher powers of AM.

There are many other ingenious propositions, although these are perhaps not of such general interest as those that have already been given. But they all go to show how far above the ordinary

the genius of Barrow was, especially when we remember how short was Barrow's professional connection with mathematics, and the relatively large and varied amount of matter that came from him in this time.

For instance he proves that, if ZD+AD is constant, then $ZD^m.AD^{m-2n}$ is a maximum, when $ZD:AD = m:m-2n$.

The proof of this theorem is generally ascribed to Cardinal Ricci, who published it in 1666. Remembering that these lectures were given in 1664-5-6, there is at least a doubt whether Barrow had not anticipated him. Even if he did not, Ricci's proof is made to depend on a lemma that if a magnitude is divided into r equal parts, their continued product is greater than that obtained by dividing it into r parts in any other manner. Barrow deduces it as an easy and immediate consequence of his theorem on a tangent to a paraboliform already quoted; so that Barrow's proof is inde-

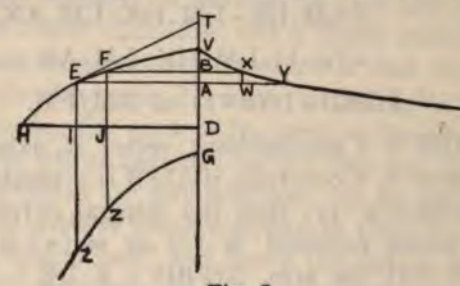

Fig. 8.

pendent of Ricci. Barrow also shows that $ZD^m.AD^{2n-m}$, where $2n > m$, is a minimum under similar circumstances.

Again, he shows, by means of his beloved paraboliforms, that if AB is the arc of a circle whose center is C, and BD is drawn perpendicular to the radius AC, then the arc AB lies between

$(3CA.DB)/(2CA+CD)$ and $(2CA.DB+CD.DB)/CA+2CD)$;

hence, taking the arc to subtend 30 degrees and the radius of the circle to be 113, he finds that the limits of the semi-circumference are 355+ and 355−; thus verifying in a rigid manner the ratio 355/113 or $3\frac{16}{113}$, which was found by Metius in the 16th century, by an unjustifiable but fairly obvious manipulation of the two limits $3\frac{15}{106}$ and $3\frac{17}{120}$. In the course of proving the preliminary lemmas for the geometrical limits given above, Barrow in effect integrates the function $a.\cos^{-1}x/a$.

Another striking instance of Barrow's (shall I call it con-

tributory laziness?) is the omission of the proof of the theorem of Lecture XI, § 27.

"Let VEH be any curve, whose axis is VD and base DH, and let any straight line ET touch it; draw EA parallel to HD. Also let GZZ be another curve such that, when any straight line EZ is drawn from E parallel to VD cutting the base HD in I and the curve GZZ in Z, and a straight line of given length R is taken; then at all times $DA^2 : R^2 = DT : IZ$.

"Then $DA : AE = R^2$: space DGZI."

The omitted proof would have run as follows:

Let VXY be a curve such that, if EA produced meets it in Y, then $EA : AD = AY : R$. Divide the arc EV into an infinite number of parts at F, M, etc. and draw FBX, MCX, etc. parallel to HD, meeting VD in B, C, etc. and the curve VXY in the points X; also draw FJZ, MKZ, etc. meeting HD in J, K, etc. and the curve GZZ in the points Z.

Then $AY.AD.BD = R.EA.BD = R.(EA.AD + EA.AB)$,

and $BX.AD.BD = R.FB.AD = R.(EA.AD - IJ.AD)$;

hence, if XW, drawn parallel to VD, cuts AY in W, we have

$$WY.AD^2 = WY.AD.BD = R.(EA.AB + IJ.AD).$$

But, as in previous theorems, $EA : AT = IJ : AB$, $AB.AE = AT.IJ$;

$$WY.AD^2 = R.(AT.IJ + IJ.AD) = R.DT.IJ.$$

Now $DA^2 : R^2 = DT : IZ = DT.IJ : IZ.IJ$;

$$R^2 : IZ.IJ = AD^2 : DT.IJ = R : WY.$$

Hence, since the sum of the rectangles IZ.IJ only differs in the least degree from the space DGZI, and the sum of the lengths WY is AY; it follows immediately that

$$R^2 : \text{space DGZI} = R : AY = DA : AE.$$

The important points about this theorem are

1. that Barrow says "Perhaps at some time or other the following theorem, deduced from what has gone before, will be of service; *it has been so to me repeatedly*";

2. that, if DT and DH are taken as the coordinate axes, and it is taken into account that the tangent ET makes an obtuse angle

with the x-axis, then $DT = x - y\,dx/dy$; also $IJ = dy$, and WY is $d(y/x)$. Hence the analytical equivalent of the equality

$$WY.AD^2 = R.DT.IJ \text{ is } Rx^2.d(y/x) = R.(x - y\,dx/dy)dy;$$

or

$$d(y/x) = (x\,dy - y\,dx)/x^2.$$

Thus Barrow had the geometrical equivalent of the *differentiation of a quotient*, and found it of service repeatedly.

I will make one more quotation. As an example of a method of construction given for drawing, in general, curves such as the one given below, we have the following:

"Let AEG be a curve whose axis is RAD, such that, when through any point E taken in it a straight line EDM is drawn perpendicular to AD, and AE is joined, then AE is always a mean proportional between a given length AR and AP, of the order whose exponent is n/m. It is required to find the curve AMB of which the tangent at M is parallel to AE.

"I note, about the curve AM, that n : m = AE : arc AM.

"If n/m = 1/2 (or AE is the simple geometrical mean between AR and AP), then, AEG being a circle, AMB is the primary cycloid. Hence the measurement of the latter comes out of a general rule."

Thus Barrow obtains the fact that the arc AM of a cycloid is twice the corresponding chord of the circle. Most of the theorems on the cycloid are due to Pascal; but in the *Encyclopaedia Britannica* the rectification of the cycloid is ascribed to Wren. If the reference there given to the *Phil. Trans.* of 1673 is correct, it follows that Wren was anticipated by Barrow. It is well known that previously only one curve, the semi-cubical parabola, had been rectified.

Lastly it may be noted that many of Barrow's theorems in *Lectio* XI, when translated into analytical form, are nothing more or less than theorems on the change of the independent variable in integration. Thus he shows that

$$\int y\,dx = \int y/(dy/dx)dy, \quad \int r^2 d\theta = \int r^2(d\theta/dr)dr.$$

Many other points might be made, but, in Barrow's words,

Haec sufficere videntur.

The two points now remaining to be considered are:

1. Why, if Barrow's genius and knowledge were so great, did he not complete the work he had so ably begun, and be hailed universally as the real originator of the calculus?

2. What influence did his predecessors have on Barrow, and what influence did Barrow and Newton have upon one another?

On the question as to the sources from which Barrow derived his ideas, there is some difficulty in deciding; and the narrowness of my reading makes me diffident in writing anything that might be considered dogmatic on this point; so that the following remarks are put forward more or less in the fashion of suggestions.

The general opinion would seem to be that Barrow was a mere improver on Fermat. But if we are to believe in Barrow's honesty the source of his ideas could not have been the work of Fermat. For Barrow religiously gives references to the ancient and contemporary mathematicians whose work he quotes. These include Cartesius, Hugenius, Galilaeus, Gregorius a St. Vincentio, Gregorius Aberd. (James Gregory of Aberdeen; in connection with this name, Barrow makes the noteworthy statement that he does not care to put his "*sickle into another man's harvest*"—the reference being to Gregory's work on evolutes and involutes), Euclides, Aristoteles, Apollonius and many others; but no mention is made of Fermat, nor does he use Fermat's method of determining the tangent by a maximum or minimum ordinate. On the other hand he may have deliberately omitted reference to Fermat, because his criterion of tangency of lines and curves was so similar to this method, that he might have provoked by the reference accusations of plagiarism. There is a distinct admiration shown for the work of Galileo, and the idea of time as the independent variable obsesses the first few lectures, an idea which he evidently obtained in the first place from Galileo, as did Newton also. But, like Newton, he simply intends this as a criterion by means of which he can be sure that one of his variables shall increase uniformly. Also, we learn from the preface that these preliminary chapters, in which he discusses time, were an afterthought; Barrow says "falling in with his (*Librarius*—the publisher, query Collins) wishes, I will not say unwillingly, I added the first five lectures."

The mental picture that I form of Barrow is that of the teacher, who has to deliver lessons on a subject, reading up everything he can lay his hands on, and then pugnaciously deciding that, although most of it is very good stuff, yet he can and will "go one better." In the course of his work he happens on the paraboliforms, perceives their usefulness, and is immediately led on to the great discovery of the "differential triangle" method. I think if any one

compares the figures used, (i) for the proof of tangency in the case of the paraboliforms, and (ii) for the infinitesimal method, he will no longer inquire for the source from which Barrow got his ideas.

Personally I have not the slightest doubt that it was a flash of inspiration suggested by the former figure (indeed it was this resemblance which caused me to put into analytical form the theorem chosen as example 2 above, and led me on to the translation of the whole work); it was Barrow's luck to have first of all had occasion to draw that figure, and secondly to have had the genius to have noticed its significance and to be able to follow up the clue thus afforded. As further corroborative evidence that Barrow's ideas were in great part his own creations we have the facts that he was alone in considering a curve as a collection of indefinitely short straight lines, and that, as he states in one place, he could not see any difference between indefinitely narrow rectangles and straight lines as the constituent parts of an area.

The answer to the question as to why Barrow did not com-

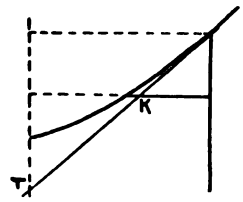

Fig. 9(i).

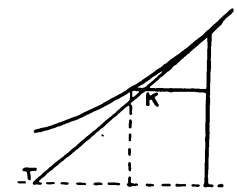

Fig. 9(ii).

plete the work he had begun is, I think, inseparably bound up with his connection with Newton; and I can imagine that Barrow's interest, as a confirmed geometer, would have been first really aroused by Newton's poor show in his scholarship paper on Euclid, for which Barrow was the examiner. This was in April, 1664, the year of the delivery of Barrow's first lectures as Lucasian professor, and, according to Newton's own words, just about the time that he (Newton) discovered his method of infinite series, led thereto by his reading of the work of Wallis and Descartes. Newton doubtless attended these lectures of Barrow, and the probability is that he would have shown to Barrow his work on infinite series (this seems to have been the custom of the time, for it is on record that Newton five years later, in 1669, communicated to Collins, through Barrow, a compendium of his method of fluxions). Bar-

row would be struck with the incongruity of a man of Newton's ability not appreciating Euclid; at the same time the one great mind would be drawn to the other, and the connection thus begun would inevitably have developed. Here we must consider that Barrow was professor of Greek from 1660 to 1662, then professor of geometry at Gresham College from 1662 to 1664, and Lucasian professor from 1664 to 1669; that Newton was in residence as a member of Trinity College from 1661 until he was forced from Cambridge by the plague in the summer of 1665; that, from manuscript notes in Newton's handwriting, it was probably during this enforced absence from Cambridge (and Barrow) that he began to develop his method of fluxions. From these dates I argue that Barrow most probably developed his geometrical work from researches begun for the necessities of lectures at Gresham College in the years 1662-3-4, and further elaborated them in the years 1664-5-6; that Newton would have not only heard these lectures before he had to leave Cambridge, but also would have had the manuscript to read, as a loan to a pupil from a master who had begun to take a strong interest in him; and that thus Newton would have got the germ of the idea from Barrow, but that the accident of the forced disconnection at this time made Newton follow the idea up in the manner and style which was essentially his own.

The similarity of the two methods of Barrow and Newton is far too close to admit of them being anything else but the outcome of one single idea. For the fluxional method the procedure is as follows:

1. Substitute $x+\dot{x}o$ for x and $y+\dot{y}o$ for y in the given equation connecting the fluents x and y.

2. Subtract the original equation and divide through by o.

3. Regard o as an evanescent quantity, and neglect o and its powers.

Barrow's rules are, altered in order for the sake of the correspondence:

2. After the equation has been formed (Newton's rule 1) reject all terms consisting of letters denoting constant or determined quantities or terms which do not contain a or e (which are equivalent to Newton's $\dot{y}o$ and $\dot{x}o$ respectively); for these terms brought over to one side of the equation will always be equal to zero (Newton's rule 2, first part).

1. In the calculation omit all terms containing a power of a or e,

or products of these, for these are of no value (Newton's rule 2, second part, and rule 3).

3. Now substitute *m*, the ordinate, for *a*, and *t*, the subtangent, for *e*. This corresponds to Newton's next step, the obtaining of the ratio $\dot{x}:\dot{y}$, which is exactly the same as Barrow's $e:a$.

The only difference is that Barrow's way is more suitable to his geometrical purpose of finding the "quantity of the subtangent," and Newton's method is peculiarly adapted for analysis, especially in problems on motion. It is particularly to be observed that Barrow, in giving a description of his way, writes throughout in the first person singular. Although at the time of publication of the lectures Barrow had seen the fluxional method, or "a compendium" of it, as it passed through his hands on its way to Collins, yet he left his own method as it stood; probably he used it freely (he applies to it the words *usitatum a nobis*—the word *usitatum* being elsewhere written to denote *familiar* or *well known*; also mark Barrow's use of the more or less usual plural *nobis* in opposition to the first person singular when describing the method) to obtain hints for his tangent propositions, but not thinking much of it as a method compared with a strictly geometrical method, probably because he could not always find a geometrical construction to correspond; yet he admits it into his work "on the advice of a friend" on account of its generality. On the other hand Newton perceives the immense possibilities of the analytical methods introduced by Descartes, and develops the idea on his own lines, possibly owing to the accident of his being removed from the influence of Barrow for a short time.

There is however another possibility. In the preface we read that "as delicate mothers are wont, I committed to the foster care of friends, not unwillingly, my discarded child".... These two friends Barrow mentions by name, "*Isaac Newton*.....(a man of exceptional ability and remarkable skill) has revised the proof, warning me of many matters to be corrected, and adding some things of his own work"*..... "*John Collins* has attended to the publication." It is just possible that Newton showed Barrow the idea of his fluxional method before he had developed it fully, and that Barrow developed it in some small degree as a tool for the purpose mentioned above, and inserted it into his work. At any rate it seems to be fairly plain that Newton was the friend on whose

* Most probably in the *Optics*.

advice the method was inserted. I think however that the more probable alternative, judging from the later work of Newton, is that first given. This would explain the lack of what I have endeavored to make out to be the true appreciation of Barrow's genius. Barrow saw that the correct development of his idea was on purely analytical lines, he recognized his own disability in this direction and the peculiar aptness of Newton's genius for the task; and the growing desire to forsake mathematics for divinity made him only too willing to hand over his discarded child to the foster care of Newton and Collins "to be led out and set forth as might seem good to them," as he says in his preface. Who can tell what might have appeared in a second edition, "revised and enlarged," if Barrow, on his return to Cambridge as Master of Trinity and afterwards Vice-Chancellor, had had the energy to make one; or if Newton had made a treatise of it instead of a book of "Scholastic Lectures," as Barrow warns his readers that it is? But Barrow died two years later, and Newton was far too occupied with other matters.

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[Note.—Since writing the above article, the author has found that the *Lectiones Geometricae* form a perfect calculus. This will be explained in a forthcoming volume of the *Open Court Classics of Science and Philosophy*. —ED.]

POLYXENA CHRISTIANA.*

A REVIEW OF BOUSSET'S "KYRIOS CHRISTOS."

"But she, though dying, none the less
Great forethought took, in seemly wise to fall."
—Eur., *Hek.*, 568f.

By odds the most imposing and important apologetic of recent years is the deep-learned, deep-felt and deep-thoughted *Kyrios Christos* of Prof. Wilhelm Bousset, well known by his *Religion des Judentums*, his *Offenbarung Johannis*, his *Hauptprobleme der Gnosis*, and as editor with Wilhelm Heitmüller of the *Theologische*

*This review, written in the first half of the year 1914, has been withheld from the press thus far, along with several other such essays, in the hope that after the cessation of hostilities in Europe it might more readily "fit audience find, though few"; but the coming of such a season seems now too likely to be indefinitely delayed.

Rundschau. True, it is in many ways a questionable service this large-minded and high-hearted scholar has rendered the cause of historicism, a "sad relief" like that brought the Briton by "the blue-eyed Saxon" of old. Even Bacon seems to view it askance, with suspicious eye, and Bousset himself foresees that his own "theses" will be held to "dissolve with Drews and B. W. Smith the person and Gospel of Jesus" (p. xv),—where the double inversion[1] is exceedingly rhetorical. Yet he holds that his "book is a continuous refutation of their theses" (xv)! Certainly the volume is a weighty one, most interesting, instructive and worthy of careful study. It teems with the most valuable truth and is in general informed by a spirit of great modesty, honesty and conscientiousness. However, in spite of all these and other excellences, the book fails entirely at certain critical and decisive points to yield the "continuous refutation" as which it is offered to the world. The nature of this failure it is not hard to make clear in general terms; a detailed examination such as the work deserves, as it would be a pleasure to give, and as would be entirely convincing, would call for several such papers as the present.

What then is the stately fabric of thought reared by the Göttingen professor? What sea-wall would he heave up against the rising tide of radical criticism? Bousset attempts a genetic reconstruction of the elements of proto-Christian doctrine, a restoration and rational exhibition of the original historic process through which the early Christian mind was carried from the days of the *Urgemeinde,* the first Church in Jerusalem, down to the great catholicizer, the heresy-hunting Irenæus. By rehabilitating this process more carefully, more systematically, more thoroughly, with greater learning and with higher plausibility than any one has done heretofore,—but more especially by reforming the whole front of the Liberal criticism, by abandoning stronghold after stronghold and advancing boldly forward to the radical positions and assuming them quite as if they were his own,—Bousset would persuade his readers that since all these things *may* have happened this way, therefore they *must* have happened and surely *did* happen just this way, that so did Christianity come into being. Now, to begin with, here is a logical lapse: the very most he could thus attain would be a more or less satisfactory theory, developed from the *hypothesis*

[1] Compare the words of A. Schweitzer in his *Leben-Jesu-Forschung* (p. 490): *Drews, wie seinem grossen Meister Smith.*

of an historical Jesus. But no such theory, even though far more satisfactory than our author's, could ever prove or verify the hypothesis; to do this latter he must not only show that his theory is perfectly satisfactory, that it explains all the facts in the case, but he must also show that no other theory developed from the opposite hypothesis either does or can explain all the facts in a manner equally satisfactory. Until he does this, it is quite impossible to convert his *may be* into a *must be*; and yet it is precisely this conversion that is absolutely essential to his argument. It is a more or less clear perception of this state of case that now leads discerning German critics to admit that the historicity of Jesus "cannot be proved," that it is at best "altogether probable" (*überaus wahrscheinlich*).

Now Bousset has made no effort whatever to meet these unescapable logical demands; hence his whole elaborate structure is swung in the air. The radical holds that *everything* so carefully explained by Bousset on the hypothesis of historicity may be explained fully and in fact far more readily on the hypothesis of the non-historicity; and until Professor Bousset takes this fact into account, all his learning and patience and constructive ingenuity are of little logical avail.

This is not nearly all, however. It is not enough to consider the facts, no matter how many nor how important, that may be readily explained on a certain theory; it is absolutely necessary to consider the facts that are *hard* to explain. It is precisely these that form the proper tests; to slight or to shunt them is to abandon scientific procedure. Now there is a host of facts assembled in *Der vorchristliche Jesus* and *Ecce Deus* that are admittedly very hard to fit into any theory of an historic Jesus; it becomes then the bounden logical duty of Bousset to consider these facts above all others, not one nor several nor many, but all of them, for all of them must be explicable on his hypothesis, *if it be correct*; not one can be excepted. The notion that by ingeniously ordering a great many other more tractable data, one may evade the logical necessity of fairly meeting and managing these seemingly unmanageable data,—this notion, no matter in what high quarters nor how zealously it may be cherished, —this notion is a delusion and a snare.

Such general considerations show plainly that our author has not fulfilled the logical requirements of the situation. At this point, though they cannot outbid him in other great qualities, such men as Schmiedel and Klostermann have shown a keener and surer

sense. Of these the former has seen clearly that such paths as Bousset's cannot conduct to the goal, that there *must* be discovered certain facts that can be explained on the hypothesis of historicity and cannot be explained on any other. This is exact science. There is no other way by which "the historical character of Jesus" can be saved. Schmiedel thought he had discovered nine such data and named them not inappropriately the Nine Pillars, and his disciples have thought to widen the sacred ring. But alas! this discovery has not been confirmed. The pillars are not such granitic facts as he supposed; at the touch of criticism they crumble, they have been abandoned even by historicists themselves. Klostermann admits that appeal to them is vain, that "new and doughtier weapons will have to be forged."

But it is not only such general logical dereliction that vitiates the thought-process in *Kyrios Christos*. Flaws scarcely less serious run this way and that, throughout its structure. Let us take some examples. *Tacit assumption* abounds in this work. The author speaks regularly of "Jesus of Nazareth," thereby assuming the historical character. Yet he must know that the better phrase is "Jesus the Nazarean," and that this adjective has, at least apparently, naught to do with Nazareth. "Of Nazareth" is merely a false interpretation of *Nazaraios*, which such a critic as Bousset cannot countenance. At this point it is enough to refer to such as Oort, Friedländer, Burkitt, Abbott, Soltau, Vollmer, Burrage, and others.

Again, Bousset begins very properly with Jesus in the faith of the *Urgemeinde* (primitive congregation), which he calls Palestinian and locates definitely in Jerusalem (*die Gemeinde in Jerusalem*). Herewith he quietly assumes nearly everything. Who knows that this *Urgemeinde* was in Jerusalem? And how does he know it? From the first chapters of Acts? But Bousset himself rejects these repeatedly and decisively as unauthentic. Even Moffatt admits that the trustworthiness "rises" as the story advances. What is therein more pretentiously accurate than the account of Paul as persecutor? Yet Bousset assures us that it is all fiction. "By no means (*nicht einmal*) is it sure that Paul himself was concerned in the persecution at Jerusalem" (p. 92), though Wendt could declare he was its soul! The story in Acts ix. 1 ff. "bears the brand of the unhistorical plain on its brow" (p. 92). *Such was the contention in "Der vorchristliche Jesus,"* p. 26 f. Since in Acts we are dealing so largely with free creations and "not any way authentic documents"

(p. 97), all reason for placing the *Urgemeinde* in Jerusalem vanishes. But the immovable reasons against it remain, some of which have already been set forth in *Der vorchristliche Jesus* (pp. 24 ff.). The only natural thing for the Disciples to do after the crucifixion (if there was any) was to return to Galilee, and the oldest account represents them as so doing (Mark. xvi. 7; Matt. xxviii. 10, 16-20). The contradiction of Luke (xxiv. 47-53; Acts i. 4-8, 12 ff.; ii. 5, 14) is perfectly open, deliberate and intentional, and has a definite aim, to represent the propaganda as emerging from Jerusalem, *against the facts in the case.* Only think how utterly absurd! A few *Galilean* peasants beginning in *Jerusalem* a campaign for the deification of a man that had just been crucified *in Jerusalem*! How did these few fanatics support themselves in the midst of the crucifiers? Even at a very low cost for living they must have had some little bread—where did they get it in the midst of contemptuous enemies? What madmen to begin to preach Jesus as a God there in *Jerusalem,* where he had never done any mighty work, where his cause, whatever it was, had gone utterly and instantly to wreck! If Jesus were really a God-Man, if he really left his grave and rose from the dead and appeared to his disciples and endued them with supernatural power from on high (as the orthodox logically maintain), then such a course might seem in itself possible, though still sharply contradicting the Gospels and the oldest tradition; but Bousset accepts not one of these allegations, he denies them one and all, and so must explain not merely the contradiction of tradition but also the incredible folly, the downright impossibility of the disciples' stay in Jerusalem. This he does not do, this he makes no attempt to do. No! The idea that the *Urgemeinde* was in Jerusalem is entirely baseless and defiant of common sense.[2]

Bousset himself must have felt the error of his thought at this point, for he writes very rationally about the proto-Christian Gentile church, justly recognizing it as one of "the weightiest of established facts" that the Gentile Christian church neither began with

[2] But even if correct it would not help historicism in the least. For of whom could the church have consisted? Surely not of Jerusalemites. Without amazing miracles they could not be converted, as the author of Acts clearly perceived. But if of Galileans, then the maintenance of the church becomes unintelligible, and the sudden spread in *two years* over the world (see p. 294) becomes incomprehensible and inconceivable. Think of a few *Galileans* in Jerusalem successfully preaching the Gospel of a Crucified and Risen and Deified Jesus and spreading it instantly over all the earth! Here we have an illustration of Bousset's characteristic method; he yields so much of the Liberal position to the Radicals that the little he would retain is no longer tenable.

Paul nor was determined by him, neither at Antioch nor at Rome nor elsewhere. "The full stream of the new universal religious movement was already at flood when Paul entered on his work, and he also was at first upborne by this stream" (p. 93). This is what was expressed far less picturesquely in *Der vorchristliche Jesus* (pp. 24 f., 28, etc.) by the multifocal origin of the early propaganda. One is delighted to find Bousset again in such full accord. Now remember that Paul's conversion is placed by Wendt at the very extremest date as only six years after the beginning, in the year 35, the crucifixion being placed in 29 A. D. Remember, too, that Deissmann's new Gallio inscription brings Paul to Corinth early in 50 A. D. instead of 53, as heretofore assumed, which reduces these six years to *three*. Remember also that Bousset places Paul's persecution in Damascus, where then there must have been a Christian congregation. So then we have the "universal religious movement" and the heathen mission flooding the world (*flutete*) at the very most within three years after the crucifixion and quite independently of Paul! In all of this Bousset, gladly agreeing with Heitmüller and sadly confirming *Der vorchristliche Jesus*, is entirely right, but how shall we reconcile it with the notion of an historic Jesus who (according to Harnack) had no notion of any world-mission, how with the notion of a narrow intensely Judaic *Urgemeinde* in Jerusalem, of whom Harnack says, "crushed by the letter of Jesus they died a lingering death"? What critic has attempted, what critic will attempt any reconciliation? We need not go beyond Bousset's own pages to find the final refutation of his contention. No! the proto-Christian movement did not issue from Jerusalem, it issued from the Jewish Diaspora, from the midst of the Hellenists. As Bousset himself recognizes, the representation in Acts is fictive on its face, and herewith the central pillar of the historistic theory crumbles into dust.

Once more, Bousset finds that the pivot of the Christology of the *Urgemeinde* was the conception of "Jesus as the Messiah-Son-of-Man" (*Jesus der Messias-Menschensohn*). Rejecting the notion that Jesus called himself the Son-of-Man, Bousset distinguishes two ideas concerning this Messiah-Son-of-Man; one of a Messiah, a "David's son," a more or less wonderful man; the other of a strictly "overearthly being, heavenly, spiritual, preexistent." *It is only as this latter* that Jesus appears in the earliest known faith of the *Urgemeinde*. Bousset is very cautious but nevertheless explicit.

"So soon as the Symbol in Daniel was interpreted messianically, just so soon must the Messiah become an overearthly figure" (p. 16. Cf. *Der vorchristliche Jesus*, p. 89). Nor can Bousset point to any stage in the primal faith at which this exaltation had not taken place; so far as we can see or know, *from the very first* Jesus was so conceived as the supramundane Son-of-Man. Here, as so often elsewhere, Bousset's words are worth quoting: "It may indeed (*es mag wohl*) in the beginning have been the prevailing opinion that Jesus as simple man (παῖς θεοῦ) walked here upon earth and was exalted (*erhöht sei*) to be Son-of-Man only after the end of his life. But certainly (*freilich*) the time is not at all distant (*gar nicht fern*) when Jesus will become out and out (*ganz und gar*) a heavenly spiritual being preexistent and descended from above" (p. 19). *"Es mag wohl"!* This sop to Cerberus was necessary. Surely it is tiny and wizen enough, what greed could grudge it? But such a "prevailing opinion" has nowhere a basis in tradition or in fact, its *problematic existence* is only an *inference* from the false assumed premiss of the pure humanity of Jesus. That any such opinion could have undergone any such "rapid" transformation, that the crucified Rabbi should have been transfigured almost instantly into a God, indeed into *the* God and made everywhere the Lord,—in Palestine, in Jerusalem, in far off and widely separated heathen capitals,—and the center of "the monotheistic Cult of the Jesus" (Deissmann), this is incredible if anything can be, and neither Bousset nor any of his peers has any explanation to offer. It is here, as elsewhere, that Bousset by his concessions (as his German reviewers complain) has given away the whole lost cause of historicism.

Such reflections as the foregoing are aroused from page to page of this great work, but we must hurry on. In *Ecce Deus* a section is given to the epithet Lord (*Kyrios*) as applied to Jesus, and it is argued that the early use of this term indiscriminately to denote both the Jehovah of the Old Testament and the Jesus of the New indicates clearly that the Jesus must have been thought as in some sense Jehovah, not perhaps as absolutely identically God but as representing the godhead in some vague way as an aspect or person thereof. This argument seems to stand yet unbroken in strength. Bousset seeks apparently to turn its edge by a very thorough study of the use of the term *Kyrios*. He finds it comparatively rare in the Gospels and Acts, much more frequent in the Paulines, and

concludes that it is characteristic of the Gentile church and derived not from Cæsar-worship, but from the heathen cults with which the church was surrounded, but he is careful to concede its regular use in this church from the very start. Now it might be granted that the example of the heathen cults around, with their Lord Osiris, Lord Sarapis, and the like, may have given occasion to the Gentile Christians to speak of their Lord Jesus. The question, however, is not, how did they come to use the term, but rather, how *could* they use it of a mere man, however exalted, or even of a supernatural being not in some wise identified with Jehovah, the Lord of the Old Testament? For it is well known that the early Christians were familiar with the Hebrew Scriptures in the Septuagint or other translation, that they recognized these Scriptures as the highest if not the only authorities, and that Lord (*Kyrios*) therein is the peculiar appellation of Jehovah, the supreme God. No matter, then, what the abandoned heathen cults might say, the Gentile Christians could not but know that Lord (*Kyrios*) meant the highest God, and it remains as hard as ever to see how they could use the term both of Jehovah and of Jesus (often indiscriminately), unless they in some manner or measure identified the two. While then Bousset's investigation is interesting and valuable, it merely answers a collateral question and leaves the original argument as well as the original difficulty untouched.

One other point. In accumulating instances of the term Lord applied to the center of a cult, it is noteworthy to find all are gods and not men, with one sole apparent exception, which Bousset pushes to the front, that of Simon and Helena: "Hippolytus reports of the followers of Simon Magus that they reverence Simon in the form of Zeus, Helena in the form of Athena, him calling Lord and her Lady. This very interesting notice is expressly confirmed by the representation of the pseudo-Clementine Homilies" (p. 117). Both these statements lack warrant. What Hippolytus says is that "they have an image both of Simon in form of Zeus and of Helena in form of Athena, and these (images or forms) they worship, him calling Lord but her Lady. But if any one by name shall call among them, having seen, the images whether of Simon or of Helena, an offcast he becomes as unknowing the mysteries." Whence it appears that they worshiped these images in form of Zeus and Athena not at all as Simon and Helena, but in all likelihood as symbols of mysterious powers of nature and of thought, and the charge that they were

worshiping Simon and Helena is merely one of many silly slanders that Bousset should not encourage. This view, and not the one quoted from Bousset, is confirmed by the Homilies, where we read (II, 25) that Simon "says he has brought down this Helena from the highest heavens to the world, this Helena being Lady (*Kyrian*) as all-Mother, substance and wisdom....for she who is really the truth was then with God supreme." How little such high theosophy offended the early Christians is seen in the immediately following statement that "we (Aquila and his fellow Christians) were his (Simon's) fellow workers at first." Again (XVIII, 2): "We (Peter and Christians) do not hold, Simon, that from the mighty power also the Lordly (*Kyrias*) called, proceeded two angels, etc." Whence it appears as clearly as we could hope that the feminine form (*Kyria*) is used only because it refers to feminine nouns, abstractions, such as Power, Substance, Truth, Wisdom. Simon may have tried to explain the myth of Helen (as in fact is said in II, 25) in terms of these concepts, but to think of these Simonians (*an early name for Christian*,—Orig., *Con. Cel.*, V, 62) as worshiping Simon and Helena, is a conceit that blots the page of Bousset. Lastly, the clause "him calling Lord but her Lady" is simply a pious invention of Hippolytus, of course, "for the greater glory of God." The words are not in Irenæus (I, xvi, 3, Harvey), from whom this good Bishop is quoting. It is in fact almost too well known for statement that the Catholic representation of Simon is simply an atrocious slander, to which Harnack lends no sanction whatever, declaring Simon to have been "the counterpart of Jesus," who made "an attempt to create a universal religion of the highest God," of whom "the later tradition is the most distorted and tendentious conceivable." That this great monotheist Simon is the original of the Gospel Simon, into whom he has been transformed in Christian tradition under the name of Peter, is a proposition I have maintained for twelve years with unshaken confidence, without finding leisure for its open discussion. It seems very late in the day to remark that the whole legend of Simon, especially of his carrying round with him a harlot Helen, is an utterly scandalous libel, always with some a favorite form of argument. This one triple question, however, I would submit to critics who have some sense of depth, of a third dimension, in construing old Christian scriptures:

1. Is it possible to read Acts viii. 4-25, particularly 13, in connection with Origen, *Con. Cel.*, V, 62, and the whole Simonian legend,

especially such words as those quoted from Aquila, without feeling that *Simon Magus was a proto-Christian*, that he stood in some close and vital connection with the early propaganda, however he may have fallen later into disrepute?

2. Is it possible to read the Gospel story of Simon Peter, of his trying to walk on the water and failing, of his being rebuked as Satan, as a scandal, as minding not things of God but of man (Matt. xvi. 23, compared with Acts viii. 20-24), of Satan's desire to sift him (Luke xxii. 31), of his denial of Jesus, of his rebuke by the Risen One (John xxi. 15-23), of his (Cephas's) crookedness at Antioch (Gal. ii. 12),—is it possible to read all this in connection with the Simonian legend and not feel that Simon Peter also had much to his discredit in early Christian tradition, that he was most conspicuous as a proto-Christian leader, and yet that his antecedents left a great deal to be desired, and that it was not possible to set him forth as a genuine unwavering disciple of the Jesus?

3. Lastly, can it be an accident that the Fourth Evangelist so studiously relegates Simon to a secondary place, that he declares three times for no apparent reason and with no apparent ground that Iscariot was Simon's son, that he represents Peter as abandoning the cause and returning to his earlier craft ("I go a-fishing", xxi. 3), that he declares three times that Peter was "standing" (xviii. 16, 18, 25), although he must have known that the Synoptics declared he was "sitting" and that "standing" was the fixed and recognized epithet of Simon Magus? If all such indications be misleading, and all such coincidences mere chance, then farewell to the interpretation of documents and to the doctrine of probability.

In dealing with "the empty grave" and the resurrection, Bousset appears at his best. The former is dismissed like the snakes of Ireland—there was none. We are told that the resurrection was really the exaltation (*Erhöhung*), the installation of Jesus-Son-of-Man at the right hand of the Majesty on high, that it had naught to do with any resuscitation of the Crucified. In Phil. ii. 6 ff., Paul makes mention not at all of any resurrection, but only of the exaltation, which alone is emphasized in the John's Gospel also, where "rise from the dead" (xx. 9) and "when therefore he was risen from the dead" (ii. 22) are recognized as secondary additaments. With fine analysis this notion of the exaltation of Jesus is traced through the growing Scriptures, until finally "the belief in the exaltation of the Son-of-Man took the more concrete form, that he arose on

the third day bodily from the grave" (p. 79). This seems most excellently said and certainly correct. In the essay "Anastasis" in *Der vorchristliche Jesus* something very similar is hinted, *"Nur mit ein bischen andern Worten,"* where it is maintained distinctly and at length that the locution "God hath raised up Jesus," referred originally not to any resuscitation but to the establishment of the Jesus in power as pro-Jehovah at the right hand on high, the phrase "from the dead" being recognized as a later addition. It is highly gratifying that Bousset has attained late but independently (for he makes no mention of the essay on "Anastasis") to views so very accordant, and this fact is a very strong guarantee of their correctness. In *Der vorchristliche Jesus* emphasis was laid upon the fact that the Hebrew *qûm* in the familiar Old Testament phrase "God raised up" is translated by the exact Greek of Acts, *anestēse* (thus Acts ii. 24, "whom God hath raised up" repeats the very Septuagint words of 2 Sam. xxiii. 1, "whom the Lord raised up"); Bousset likewise says, "finally the different formulae (also the *hypsōthēnai*) may go back to the Hebrew *qûm* (Hos. vi. 2, *yqimēnû*)." On the whole, one may say, *Nun, man kommt wohl eine Strecke.*

Less satisfactory is Bousset's treatment of the general subject of "the miracle." His first and chief (though mistaken) effort is to minimize this element in the Gospel story. It seems to him a nimbus gradually thrown about the person of Jesus by the faith of the Congregation. In the earliest (Q) source it was comparatively insignificant. The Passion week also remained nearly quite free. Even in Mark some "most valuable" sections are without miracle, in others (as the first day of Jesus's activity) miracle is not in the foreground. We must always look to see whether "the interest" of the Evangelist is in the deed or in the spoken word. Still, he admits, "very early the conviction arose in the Congregation that miracles were the most important constituent of Jesus's life." So it must have been that "he did have the gift of healing, that he did cure the sick and drive out demons." Gradually tradition dipped the life of Jesus deep in the miraculous, far beyond healings and exorcisms. In some measure the Old Testament contributed to this result, which was mainly due however to the popular love of the wonderful. All sorts of marvelous stories told of others gathered round the form of Jesus, as clouds about mountain tops. Parallels may be found here and there both in Jewish and in pagan legends. From all sides miracles migrated into the life of Jesus and settled

there. In particular, the account of the Gadarene demons was at first merely a "funny story" (*lustige Geschichte*) of "poor deceived demons," but was afterward attributed to Jesus. Then there are certain "foreign bodies" also encysted in the life of Jesus, such as the transfiguration, pitched as high above as the Gadarene tale is below the ordinary level; such as the Cana miracle, which comes from the myth of Dionysos at whose temples in Naxos, Teos, Elis, such transubstantiation of water into wine was wont to take place; such, too, perhaps were the miracles of the feedings, which depict a god reigning among his people and dispensing his gifts. Like a magnet the personality of Jesus drew from all the environment all possible materials and legends to itself, where the skill of evangelic poesy fused them together so deftly that only the keen eye can recognize and discern the constituents.

Such is Bousset's diagnosis of the situation, and it might safely be left to the judgment of readers, for there are few whom it is likely to mislead. It is special pleading throughout and does no manner of justice to the most evident facts. That the Q source, a collection of sayings, should contain little reference to deeds, whether mighty or not mighty, is too natural for any comment, much less for any inference. But that the Mark source, almost if not quite as old, should be specially full of such marvels (as, generally admitted, in spite of Wendling's vivisection), is in itself a refutation of the theory of gradual accretion. Take the instance of the first day (Mark i. 14-34), to which Bousset strangely appeals as showing no main interest in the miraculous. In these twenty-one verses, Jesus calls Simon and Andrew, then James and John, all four instantly leave their nets to become fishers of men,—plainly in the meaning of Mark it is superhuman power that constrains them. Then Jesus enters the Synagogue and astounds all by his doctrine and authority,—again the deed is superhuman. There he meets a man with unclean spirit, who instantly recognizes Jesus as the Holy One of God, come to destroy such spirits. The man is cured instantly by a word of miraculous might. The people are amazed, his fame spreads instantly all abroad. Coming out of the synagogue Jesus enters Simon's house and instantly cures his wife's mother of a fever. The cure is complete, instantaneous, she rises forthwith and goes to work. At sunset all the sick and demoniac are brought to his door; he heals many and casts out many demons, and will not

let them speak, because they know him, recognize him as their destroyer.

So then it appears that this "first day" is one unbroken round of miracles, one long exhibition of superhuman might molding everything with equal ease to its will. How Mark could show any greater interest in the miraculous, it seems hard to see. The notion of the transmigration of the miraculous may be in some measure correct, but it is irrelevant even in its correctness. Doubtless a painter will and must dip his pencil into the dyes at hand, but this affects not the meaning of the picture he makes. Naturally the evangelist would draw upon the general milieu of phrase and fable, of thought and expression, for the materials and forms of his symbolism. Knowing nothing of leprosy he would not represent the sin-smitten world as a leper; having never heard of demons, he would not think of depicting the overthrow of idolatry as casting them out. But being familiar with the whole framework of contemporaneous life he did precisely as Homer and Kipling did, he boldly took whate'er he did require, no matter what it was nor where he found it. Then he molded it to his own purposes and after his own ideals. He gave it his own meaning, he filled it with his own conceptions. Such is the method of every artist in every age.

Take the example of the wedding at Cana. The Dionysian parallel has not escaped my notice. It seemed and still seems possible that the particular form of the miracle was suggested by the classic myth. But what of it? Did John tell the story of Jesus simply because it had been told of Dionysos? Impossible. Whoever he was, this John was surely deep-thoughted and desperately in earnest. While it is conceivable that he might have told an actual incident just as a mere matter of history, without reflecting and without attributing to it any significance, yet it is quite inconceivable that he would invent such an incident or extract it from the mythology he despised and affix it to his Logos-God in mere wantonness, without intending something thereby. He must then have had some meaning, and this meaning was the symbolic sense of the miracle. The appeal to Bacchus merely emphasizes the necessity of understanding the miracle as a symbolism of the author's.

Similarly with respect to the exorcism at Gadara. Even if one admitted the queer conceit that it was a "merry tale" of "poor deceived devils" (thus attributing a Teutonic consciousness to the evangelist), yet this would explain only the unessential feature of

the swine, it would leave the formidable grandeur of all the rest untouched. And why should such a "funny story" of some vagrant exorcist be decked out in such regal attire and told of the godlike "center of the cult"? Here again it seems certain the fancy of the evangelist was not merely running wild, he was not talking solely to hear himself talk, he must have been narrating either because the incident actually occurred (in which case it was certainly worth preserving) or else because he *meant* something by it, because he had an idea that he wished to set forth; in this latter case, the miracle is a symbolism—and in fact too patent to escape the open eye.

The like may be said of the transfiguration; whencesoever may have come the materials and the general features of the composition, it is clear as day that the evangelists are *thinking*, they are not idle scribblers, and their *thought* is the symbolic content of the miracle itself.

In at least one case Bousset has seen and avowed the figurative sense. He speaks of the blind-born man of John's ninth chapter as "that symbol of the Congregation, born blind and become seeing," and he interprets the phrase "they cast him out" (ix. 34) as referring to the expulsion from the Synagogue of such as confessed the Son-of-Man (p. 22). Now there are many traits in this "blind-born" that remind us of Paul, as Thomæ sets forth, and to me he seems to typify the proselyte, but it makes no difference,—the point is that Bousset recognizes here a symbol and a symbolic statement of broad facts of early Christian history. If this be found necessary in the case of this miracle, which is adorned with so many details and so much local color, how much more must it be necessary in a score of cases where the symbolic sense lies stripped and bare and unmistakable?

Bousset says naught of the cripple at Bethesda, naught of the supreme miracle wrought on Lazarus. Since he recognizes the blind-born as a symbol, can he fail to recognize these as symbols also? Does any logical principle forbid the extension of this mode of interpretation? Does not the chief methodological maxim, the Principle of Parsimony, *require* its extension to every case where it *can* be applied?

The notion that the personality of Jesus attracted to itself all manner of marvelous elements, as a magnet draws iron filings, is the merest figment of fancy. What do we *know*, what have we

any ground to believe, about this personality as historical, that suggests such an idea? Nothing whatever. But if by *Personenbild* Bousset means the personality merely as it existed in the minds of proto-Christians, then, though the thought be in a measure just, it is without pertinence. For the question arises, How did they think of him? If as a man, then what in his humanity explains the magnetic attraction? *If as a God,* then indeed the attraction may be explained, but *cui bono*? Thereby the radical theomonistic view is strongly recommended, and the liberal andromonistic theory is not strengthened but is hopelessly weakened. The notion of Bousset seems to be a kind of *last resort,* which indeed assumes everything in dispute. This in fact Bousset does openly without any semblance of proof in declaring that "the historic reality of this life offered a certain basis for this further development (of the miraculous). For it cannot be denied that Jesus in his lifetime exercised the gift of healing the sick, and that healing the sick and "driving out demons" were characteristics of his wandering life" (p. 71). But it actually is denied with daily waxing emphasis, and why not deny it? How do we know that such was the "historic actuality"? Bousset is silent, he gives no hint. He merely assumes everything to be proved. Now the fact is that this notion of Jesus as a wandering healer and exorcist is utterly impracticable and intolerable to reason. Consider only that this "historic actuality," this "wandering life," (supposedly) began quite suddenly, without any reported premonition (the birth-stories are admittedly late inventions), that it lasted only a few months, and that it ended ignominiously on the cross. Instantly then the crucified is preached everywhere round the Mediterranean as the supramundane Son-of-Man, as the Lord in heaven. What possible "gift of healing" and of exorcism can make such a course of events in any degree intelligible? Such a human personality must have been unspeakably marvelous, and his followers unspeakably silly! The fact is that the historic view supposes that early Christianity was born and developed among a widespread community of madmen, that the whole Roman Empire was at that time virtually insane, even as Binet-Sanglé has inexorably expounded in *La folie de Jésus.* But even on this wholly extravagant hypothesis the cause of historicism is still lost. For if Jesus had really been such a living miracle we should have heard something about him in contemporary history and some traces of the wondrous man would have been left on the early Christian

consciousness, whereas contemporary history so far as it exists is absolutely dumb about any such man Jesus, and not the faintest trace of his memory or human personality can be detected in the early Christian consciousness itself. Bousset admits that the consciousness of the *Urgemeinde* is not of the man Jesus but of the supramundane Son-of-Man, and that there is no sign of such a human character in the religion of Paul (p. 143). According to the historistic theory the whole of early Christian times is a period of meaningless miracles.

But even if we were willing to admit all such, the case would be just as hopeless as ever. For all of these unmeaning marvels stand in the closest connectivity with an endless web of contemporary, antecedent and succedent religious and philosophic life, from which they cannot by any violence be extricated or torn away. Now in this connection this proto-Christian life is intelligible even in its minute details in the *absence* of any such prodigious personality as historicism assumes; and it is thoroughly unintelligible in this connection, even in the broadest outlines, in the *presence* of that personality.

Once more, Bousset makes appeal to the notions current at the time as favoring the hypothesis of such a wonder-worker and explaining in large measure the Gospel story. He thinks it was a superstitious age of miracle-mongers, when anything would be believed, and that the story of Jesus is fairly in line with many others. This is a favorite defense of the modern apologist, and it calls for careful consideration, but it is wholly incohesive and crumbles at touch.

It may be granted that marvelous stories have been told in every age of nearly every very notable man. These are in general very easily explained and need rarely mislead any one. But common sense says instantly and positively that they are not in any sense in line with the New Testament miracles. In all such cases there is a more or less firmwoven web of ordinary, perfectly credible narrative, close joined with the general fabric of human history, in which the miraculous elements appear as manifestly "foreign bodies" that can be shaken out or brushed off with little jar to the main structure. The miracles do not constitute the account, they are merely adventitious, often mere playful exaggerations, and not seldom transparent symbolisms.

But in the Gospels the case is wholly different. Here the

miracle is the very essence of the whole. Jesus appears, it is true, in a double character, as a Teacher or Lawgiver and as a Wonder-Worker. But even as a Lawgiver he is hardly less miraculous. For he teaches and legislates by his own immediate and personal authority ("But I say unto you," etc.). This he can do only by divine prerogative. He speaks even as God. "The Jesus says" seems quite parallel to "Thus saith Jehovah." Everywhere in the New Testament " the word of the Lord," i. e., of Jesus, is the court of last resort, is the end of controversy. So too his miracles are the deeds of his own might and person. He never appeals to God in working them. He invokes no name, he uses no instrumentality (a few apparent exceptions count for nothing). He does everything by his own word, by his own touch, by his own omnipotence. Moreover the story of Jesus exists for this Teaching and this Doing, and for nothing else. Take away these two notions, and what remains? Practically nothing at all. Even the Passion, though late and no part of the primitive Gospel, is set forth as a divine deed, not by any means as a part of the general fabric of history, but as an inroad from without, as his own voluntary self-surrender, as an act of God. Of human historical life proper of Jesus there is nothing in the Gospels whatever. Two or three incidents (as of the arrest by his friends, of blessing the children, of Mary and Martha) are exceptions only when misinterpreted,—as already set forth in *Ecce Deus.*

It is this manifest fact, that the story of Jesus is supernatural and nothing but supernatural, essentially and unalterably, from beginning to end, that distinguishes it finally and forever from all legendary stories of historic characters, where the historic and natural alone are essential and constitutive, while the supernatural is unessential, adventitious and easily removed. Whenever poetic fancy begins to weave legends about real heroes it produces results quite different from the Marcan source, textures in which the general course of human events is closely followed, with here and there a strange or marvelous incident thrown in for its edifying or glorifying effect. Such creations of fictive fancy are the first chapters of Matthew and of Luke, and even untutored literary feeling perceives at once that we enter another atmosphere in the third chapter.

If we would learn by example how the marvelous intrudes itself into history we cannot do better than to take the case of the great

Revivalist, Apollonius of Tyana. Some have thought his career so closely parallel to that of Jesus as to illustrate it and show that it was really historic. Others have found it so marvelous as to reject Apollonius himself as a creature of fancy. But the imagined parallelism is altogether unreal, in fact, on closer scrutiny there is revealed sharper and sharper contrast.[3] The career of Apollonius is in its broad outlines, and in nearly all of its details, perfectly credible and very little remarkable. The marvelous elements are rare and trivial, they may all be removed, like moles from a face, without disturbing its main features or its general character. The biographer himself has no notion that his hero was aught but a man among men, born of a woman, and bearing the same name as his father. He claims on the whole for this hero nothing beyond extraordinary insight, foresight, and possibly occasional second sight. This Tyanean lives a hundred or at least seventy years, his career is followed from period to period, it attaches itself almost blamelessly to received history from point to point, and wherever it may seem to violate probability the explanation is close at hand. All this does not indeed quite prove the historical character of Apollonius (since one might invent a thoroughly credible history), but it does show that his biography presents no serious problem.

All this we find reversed in the case of Jesus. In the older tradition, as Corssen admits, his career is quite timeless. It attaches itself neither to month nor to year. Only in later layers is there an evident attempt to connect the story with some era in history. Nor is anything known of his antecedents or family. The accounts in Matthew and Luke are patent contradictory fictions. In Mark and John the Jesus simply appears full-fledged from the first, like Athena, and at once begins a career of miracles. Though John would humanize and sentimentalize, though he makes Pilate declare, "Behold the Man," though he strives hard to represent the Logos as become flesh, yet he does not succeed, despite his unquestionable literary and religious and philosophic genius, in producing the portrait of a divine man, nay, not even of a lovely man. Strive as he will, the features of the God still shine through the human traits,

[3] Compare Norden's *Agnostos Theos*, pp. 35 f.: "When Hierokles, the foe of Christianity, compared this work of Philostratus with the Gospels and its hero with Christ, he indeed made the refutation easy enough for Eusebius [if only Eusebius had not been Arian!]; for literary connections there were none at all, and material parallels at most only in the sense in which F. Chr. Baur loved to conceive them. But the case is wholly different when the comparison is made with Acts."

which are plainly merely painted over and form no part of the primitive sketch. Say what you please, the Johannine Jesus is not lovely, is not attractive, as a man. There is much high-wrought theology and sublime religion in his discourses, and these we may greatly and justly admire. But these are manifestly merely the musings of John and give us no notion whatever of the man Jesus. Indeed the failure of the Evangelist to depict an attractive human personality is one of the most notable in all literature. His man Jesus is at every turn remote, austere, enigmatic, often mocking, unfeeling, unintelligible, and requiring apology. This is easily proved chapter by chapter.

What is there lovable in the Jesus of the first chapter? Nothing. In the Jesus of Cana? Nay, he is even stern and unfilial, the commentators must explain away his words. What in the Jesus in the Temple, where the deed is simply of power, not of justice or love? What is there to praise in his treatment of Nicodemus, whom he merely puzzles and mocks? In iii. 22-36 the Baptist talks precisely as the Jesus, showing that it is the Evangelist that is speaking all the time. At the well, what single word or deed of kindness? None at all. It is only unearthly authority walking on earth. Nor is there aught in the following verses. The second sign, the healing of the nobleman's son, is a deed of might solely, with no traces of love or human affection. The same must be said of the case of the cripple. It is only a defiant exercise of divine power on the Sabbath, no glimmering of human emotion. Nor is there any more in the long speech that follows.

Coming now to the sixth chapter, we find the five thousand fed. Is it an act of human sympathy, kindness, self-sacrifice? By no means, but only of divine power, symbolizing the all-sustaining exhaustless might of the Truth, the doctrine of Jesus. Likewise the miracle of the ship brought instantly ashore, by might of the God. The long address that follows is doubtless profound theosophy, but it merely mystifies the hearers.

Similarly chapter vii shows no really human trait, least of all any amiable trait, it only perplexes the auditors with doctrines deep below the utmost plummet of their understanding. Chapter viii contains the famous pericope concerning the woman (vii. 53-viii. 11), which surely displays no human quality but teaches the forgiveness of God for the wicked and adulterous (i. e., idolatrous) generation, which the Jesus-cult had come to save. As a historic incident it is

not defensible, and by the early church was not felt to be defensible; the rest of the chapter is discussion between Jesus and the Jews, in which there may be much profound theologizing, which none could expect the Pharisees to understand, but certainly nothing to move any one to love.

The ninth chapter gives the symbolic miracle of healing the blind-born; there is in it never a movement of human feeling, only the enlightening power of the "monotheistic Jesus-cult" is set forth and enforced. This was well worth doing well, but it teaches nothing whatever about the gentle humanity of Jesus. Chapter x sets forth that Jesus is the Door and the Shepherd. It is all doctrine and nothing but doctrine. Even the notion of laying down and taking up life is pure dogma, set forth with utmost frigidity without any tinge of human emotion. In fact it seems clear that x. 11-18 is an appendix, which Wellhausen has perceived, as also the Latinism in "placing" and "again taking" life (*theinai* and *palin labein*). Plainly these are words of a musing dogmatist and wholly impossible on the lips of any sane being addressing the Jews. Least of all do they present the Jesus in a lovable light, since they merely bewilder his audience.

The resurrection of Lazarus is beyond question a symbolism, whether Lazarus be this or that, the Gentile world, or humanity in general, dead in trespasses and in sins. Clearly no care for Lazarus controls the conduct of Jesus, who pays no heed to the message of the sisters, but waits quietly till Lazarus dies, solely in order that the Divine power may be exhibited in his resurrection. True, it is said that Jesus loved Lazarus and Mary and Martha, precisely as it is said that "God so loved the world." Divine love does indeed seem to wait upon the slow process of the suns, but not human love. A man that would wait till the last moment before helping his friend, in order to show forth his own power more brilliantly, would arouse only abhorrence. Some traces however of human passion seem to present themselves in verses 33-35, where it is declared he "groaned in spirit and was troubled" and "Jesus wept." But "groaned" is not the right word. Far better is Weizsacker's *Ergrimmte er im Geist und schüttelte sich.* It is wrath, not grief, that is meant by *embrimāsthai* ("to snort at," as in the "snorting of Jehovah"); even Godet concedes as much (*L'évangile de Saint Jean,* III, 231). This choler is not so easy to understand, but it is a fact in this representation. Neither then can we interpret

the weeping as the expression of tender sympathy. In fact there was little room for grief—at what? Lazarus was immediately to be resuscitated for the greater glory of God. The feelings of Jesus seem to have been directed not at all toward Lazarus, Mary or Martha, but solely toward the Jews and himself in relation to the Jews and to God.

In chapter xii Jesus is purely dogmatic and self-glorifying without any tincture of altruistic feeling. In chapter xiii he washes the disciples' feet, but in no spirit of humility or devotion, solely as a symbolism, apparently to displace the symbolism of the Last Supper. The new commandment (Love one another) is new only in putting *agapate* for *phileite*; the notion and obligations of mutual love were perfectly familiar both to Jewish and to Greek ethics. To say that among Christians the word was filled with richer meaning is indeed to say, but not to prove. But even if such were the case it would mean only that in the early religious community the flame of sympathy was kindled to a livelier glow—which would require no explanation. It would call for courage to contend that the Johannine Jesus must have been a lovely character because he exhorted his disciples to love one another.

Chapters xiv, xv, xvi are theology or Christology pure and simple, with still no play of human or indeed any other feeling. They set forth the doctrine of the Jesus and nothing else. The love frequently mentioned is love divine, toward God or from God, such as a saint might feel or a sinner might receive, but it is not a love that tells anything about a Man Jesus. The same must be said of the great high-priestly prayer, chapter xvii. It is nothing whatever but Christology, dogmas concerning the Jesus-Logos and the Father and the church, taught with the highest authority because placed on the lips of Jesus under circumstances of the deepest solemnity, but it gives no hint as to any human character of the Jesus. The love mentioned is a purely theological love, as the sin denounced (xvi. 9) is purely theological, "they believe not on me"; and the judgment also (the overthrow of idolatry), "for the prince of this world hath been judged"; yea, the righteousness also, "because I go to the Father and ye behold me no more."

Chapter xviii presents the Jesus in godlike majesty, imperturbable before Annas and Caiaphas and Pilate, but still without human sentiment. Chapter xix describes the persecution and death of a God, but still with only the most insignificant touch of human-

ity. The sufferer does not really suffer, he merely plays a part in a sublime symbolic tragedy. It is to fulfil the Scripture that he says "I thirst"; then he proclaims, "It is finished," bows his head and *delivers up the spirit.* It is all quite voluntary; the nails have not slain him, no one has taken his life, he has himself given it up. The words to the mother and to the beloved disciple, "Woman. behold thy Son!" and "Behold thy mother!" breathe only the faintest breath of human sentiment. They are plainly allegorical and seem to refer to the complete passage of the "new doctrine" from the Jew over to the Gentile.

Neither after the resurrection is the human character of Jesus either more or less lovely. In all the apparitions,—to Magdalena, to the disciples, to Thomas, to the seven,—the same unearthly aloofness is present, precisely as before the crucifixion. Even in the conversation with Peter there is never a heart-beat. We are indeed told repeatedly that he loved one disciple, but there the story ends. That this disciple leaned on Jesus's breast is doubtless said symbolically, but even if said literally it would merely indicate position at the table, or at most only the fondness of the disciple, it would tell naught about the human character of Jesus.

In the foregoing no question is raised as to the unity of the Johannine text; the proof that the text has suffered extensive revision would not affect our general conclusion, but it would forcibly illustrate the all-important truth that all of our New Testament Scriptures, with very few and insignificant exceptions (if any), are gradual growths, the stratified deposits of centuries of intense religious activity.

It seems then that the Fourth Evangelist has not introduced into his portrait a single really attractive feature. As a human being his hero has not one element of loveliness; nay more, in spite of eighteen centuries of prejudice the fair-minded reader must admit in his own heart that the portrait is unlovely, that it is ghostly and uncanny, stern, harsh,[4] and unfeeling. Nevertheless the Evangelist has evidently striven hard to make the picture both human and lovely. "Behold the Man" is a clear cry from his own soul. He has imagined details without number whose only function could be to make the painting vivid and realistic; he has wrought countless

[4] Even Weidel cannot deny but has to admit this; see his *Persönlichkeit Jesu*, passim, especially p. 53 f., where he decides that not Love but Wrath, not the Mild but the Harsh, was fundamental with Jesus.

variations on the theme of love, he has studied earnestly to introduce tender and intimate relations. He has humanized to the utmost, he has sentimentalized to a degree. And yet—his failure could scarcely be more complete. From first to last, despite all these efforts, the Jesus remains a God, the same yesterday, to-day and forever, with hardly the slightest change in visage, in tone, in bearing, through twenty-one chapters, without a single warm pulse of blood, as "cold as the waveless breast of some stone Dian at thirteen."

This result is remarkable. It shows how completely possessed was the mind of the Evangelist with the notion of Jesus as Divine. With extremest care he would paint a human form and face; but nay, the humanity is only the most transparent veil, through which gleams immovably and almost mockingly the visage of Deity. The other Gospel-writers have made no such studied attempt to depict a God-man; they have indeed historized and humanized, but in a far franker and more incautious manner, with far less care for detail, with broad strokes rather than delicate pencilings. Their failure to produce a really human figure is just as complete as their successor's, though far less conspicuous and impressive because they have essayed so much less; the disparity between the endeavor and the accomplishment is not so patent and painful. It is needless to go through the Synoptics chapter by chapter. Whoever does so will find that he seeks in vain for a genuinely human trait or deed, the few apparent exceptions have been sufficiently treated in *Ecce Deus*, such as the incidents of the little children, of the Rich One, and of Mary and Martha. The writers are not concerned, as was John, to make us "behold the man," they draw their sketch much more naively, according to this or that divine model, whether the Suffering Servant of Jehovah, the Alexandrine Wisdom, or the Danielic-Enochian Son-of-Man. The point is that it is plainly a Divine being that moves before us, and not a man of flesh and blood.[5]

How enormously different is the representation here, even in the oldest strata of the Synoptics, from any depiction of any man, even of the most wonderful, is seen clearly in a single circumstance.

[5] It seems almost impossible to state the case with due emphasis. That a man Jesus, even though far below the conceits of any historicist, should not have been thoroughly lovable and intensely human, is quite incredible; and that tradition should not have preserved one single trace of the lovable while deifying him as the God of Love, seems improbable to the verge of the impossible; it remains then that he was a humanized God, but never a man at all.

Apollonius is represented, and perhaps correctly, as absolutely chaste and virginal, yet as recognizing fully and wisely the rights of Aphrodite, and no one would feel the least shock, had he been represented as falling in love or as married. But in the case of Jesus any such representation would be blood-curdling, it would be felt as blasphemously impossible. To me at least the insinuations of Renan (to say naught of Binet-Sanglé), when I first read his "Life of Jesus" sympathetically many years ago, were immeasurably loathsome, as well as ridiculous, though I never dreamed then that Jesus was aught but the man of Galilee. The fact is that we all feel the jar and dissonance even when told that Jesus was an hungred or that he slept. We see at once that these traits have been introduced solely to vivify the story in the context, yet we also feel that it is a decided artistic defect of the story that it should require any such detail to make it vivid; but surely no one has any such feeling about Apollonius or any other human being.

In general, however, in their historizing the Evangelists have avoided such pitfalls most admirably; they tell us mainly that Jesus said or went or did, with little further specification; but these vague terms were necessary in the nature of the case, they were familiar enough as predicates of Jehovah—who could evade them in any anthropomorphization or historization? Add hereto the much rarer use of certain cognates and synonyms, such as "entered," "departed," "walked," "said," "heard," "knew," "perceived," "talked," and the tale is well-nigh told. All of these and even "sat" are used of Jehovah, and not improperly but of necessity. A few other still rarer uses have been sufficiently discussed in *Ecce Deus*.

Eating would seem to be more unbecoming to the Jesus than Byron thought it was to woman, though the ancients in general deemed not so, but conceived of the gods as feasting daily on Olympus though only on heavenly food, "For no bread do they eat, nor drink of the wine in its sparkle," and making a twelve-days sojourn in feast with the blameless Aethiopes; nor was the Hebrew idea much different. It would not be strange, then, if the Evangelists should represent the Jesus as eating, yet only Luke speaks of him so and remarkably only after the resurrection (xxiv. 43)—an apparent thrust at Docetism. True, a Pharisee desires that Jesus eat with him (Luke vii. 36) and Jesus declares "with desire I have desired to eat this Passover with you" (xxii. 15), but it does not appear that he actually ate, for he adds, "I eat it not until it be fulfilled in

the kingdom of God" (Luke xxii. 16). Also directions are given to prepare "where we may eat the Passover," "where I may eat the Passover" (Mark xiv. 14, Luke xxii. 11, *not* "where I shall eat"),—"where thou mayest eat" (Mark xiv. 12), "for thee to eat" (Matt. xxvi. 17), and it is said that the disciples ate, but not the Jesus.

In John iv. 32 the idea of his eating seems to be distinctly rejected, while in the synoptics it appears to be avoided. On the whole the representation of Jesus as human is carried in the Gospel only so far as the general needs of the symbolism require, but hardly further. The vivid depiction of a striking personality is nowhere to be found; on the contrary, the notoriously unhistorical elements abound, the representation is thoroughly conventionalized and drawn after purely unhistorical and at least quasi-divine models, and the characterizations are so openly discrepant or downright contradictory as to render the task of ascertaining even a few principal features entirely hopeless. This general state of case is practically conceded by such competent critics as Bacon (*Christianity Old and New,—Characterization of Jesus*) and Weidel (*Persönlichkeit Jesu*), to name only the most recent (1914). But such as Bousset and Conybeare may say, "Is not the character of Apollonius equally uncertain?" The appeal to Apollonius grows daily louder and more insistent, and since historicism would change the venue to Tyana, to Tyana let it go. Nothing could please the present writer any better, for it is not hard to show that by this much paraded parallel to Jesus the historicity is finally and hopelessly condemned. However the question is a large one and deserves a full and separate treatment shortly to appear, much more minute than already given. Here be it noted only that Norden in *Agnostos Theos* holds firmly that Acts is dependent upon original memoirs of Apollonius.

In conclusion, what has Professor Bousset to claim for the simulacrum of an historical Jesus, which he has poured forth such wealth of learning to defend? The passage is eloquent and worth quoting in full. Not only does it show Bousset at his best, but it also shows the desperate plight of historicism even when such a shield of Apollo is spread above it in defense. It begins Homerically enough: "So has the church (*Gemeinde*) woven its poetry into the figure of its master's life. But it has done more than that and withal has preserved a good piece of the genuine and original life. [We prick up our ears in wonder, to hear the proof, but in vain,—no

gospel of which he had never dreamed, whose memory was forthwith forgotten or transfigured into its own utter unlikeness? It is clear that this "historic" figure, as utterly ineffective, is utterly useless in the interpretation of early Christian history; it is only the divine figure that works. What reason then for assuming such human figure? None at all. Hitherto it has been held that the human Jesus was necessary to explain proto-Christianity; and hence the reality of the historical character was inferred. But now it appears that this same character is quite inoperative, that only the divine Jesus was "effective." How then are we to deduce the historic actuality of Jesus, since one premise in our syllogism is gone?

But the case is really far worse. Not only is "the historic Jesus" seen to be useless as a fifth wheel, but the notion is a positive and a heavy clog. If there was any such historic character, then the formation of the supramundane Jesus-figure becomes doubly, trebly hard to understand; the consolidation, the precipitation of all the elements present in the historic milieu into the Idea of the God-Jesus seems not impossible to comprehend; but their shaping into this *divine form* when deposited on an immensely different *human form,* this seems well-nigh inconceivable. Over against any tendency of these elements to crystalize into the New Testament image of the Logos or the Son-of-Man enthroned in heaven as world-ruler, would lie the obstinate facts of memory, of ordinary earthly life, of humiliating crucifixion. Bousset makes no attempt to show how such a transformation did or could take place. Even if it were possible for such a metapsychosis to occur in the minds of a few, it seems many times impossible for it to have taken place in the minds of all. Yet it must have done so, for we find the same doctrine of the Divine Jesus on the lips of all the early missionaries, unaffected by a thousand variations in detail. Add to this that the transformation took place practically instantly; for before Paul's conversion, before the end of five years at most,—nay, the recently discovered Gallio inscription (of Deissmann) brings Paul to Corinth early in A. D. 50 instead of 53 as hitherto thought, so we must now say at most *two years,*—after the crucifixion we find the Gentile mission "flooding" the world with the doctrine of the heavenly Jesus —and not only the inutility but still more the impossibility (without a continuous psychological miracle) of the doctrine of the transformation of the Jesus-figure becomes manifest.

What Bousset says the *Gemeinde* has preserved of the original

Jesus-figure is too vague for argument. Not one of these elements can be traced back with any confidence to a personal Jesus; they are altogether as easy, yea, they are much easier to understand as the products of the general religious consciousness dominant in proto-Christian circles. Of course this consciousness came to expression only in individuals, many of whom were doubtless notable personalities, *none of whom was the God-Jesus* any more than Isaiah was Jehovah. Bousset himself seems to admit that there was naught new and peculiar therein, but holds that it was a question of fire, energy, vividness, passion and power. All this appears most true, but where, pray, are we to seek for all these elements of efficiency? In "the historic Jesus," or in the preachers of "the doctrine of the Jesus"? Common sense cannot hesitate. Assuredly it is the contagious zeal of the early missionary movement, the boundless enthusiasm of a prodigious idea, of militant monotheism, that accounts for all that Bousset rightly stresses. To this exalted religious consciousness, this transport of a sublime faith in one God among many idols, this amazing conception of a Kingdom of the Heavens, of a converted world, this inner voice, "Woe is me if I preach not the Gospel," this vision of the angel flying in mid-heaven and crying aloud, "Fear God and give him glory"—to this religious consciousness, the birth of brooding centuries, must we ascribe the high and distinctive qualities that Bousset so clearly discerns and so brilliantly sets forth.

In particular it is the elevated style of the New Testament, the *élan vital* of its religious rhetoric, that enthrals the reader and makes him exclaim, "never man spake as this man." But this supreme quality has certainly naught to do with any human personality of Jesus. It presents itself under a hundred Protean forms in the New Testament, in the Synoptics, in the Johannines, in the Paulines, in Hebrews, in the Epistles, in the Apocalypse, every instant changing and everywhere the same spirit, whether in Peter or in Paul, in John the Baptist or in John of Patmos. It is the spirit of the "new teaching," the conscious burden of the message of Salvation, the Glad Tidings of great joy, of the Redemption of Humanity from the ancient tyranny of the demons of idolatry into the Kingdom, into the freedom of the sons of God. This was undoubtedly the greatest propaganda ever proclaimed to the race of man, and it would be strange if it had not heated the furnace of religious feel-

ing to sevenfold ardor and expressed itself in a sacral literary style of peculiar energy and unction.

The proto-Christians themselves took exactly the right view of the matter, saying, "It is not ye that speak, but the spirit of your Father that speaks through you," and to the Spirit they ascribed all the mighty deeds of the Apostolic age. Such was really the case; it was the Holy Spirit, the communal religious consciousness shared by all alike but in varying measures and forms of manifestation, that inspired the "mission-sermon" of the monotheistic crusade and wrought the amazing wonder of converting an empire. If in later centuries the church has achieved no triumph commensurate with the first, the reason is simple enough; it has not been animated by any Idea comparable in sublimity or in truth with the proto-Christian Idea, the monotheization of the pagan world. No greater error than to force individual religion, the desire for personal salvation, to the front in this far-flung battle-line of missionary religion. Of course the Apostles wanted to be saved, but they felt sure they were already saved, it was the salvation of the lost that concerned them; they set up high standards of moral and religious conduct, to which they strenuously exhorted their converts, but the supreme matter was to worship God and Him alone, all the rest was secondary and sequent, even as perfectly expressed in Matt. vi. 33: "Seek first the kingdom of God and his righteousness, and all these things shall be added unto you." A deep sense of personal guilt, with longing for personal salvation, might possibly make a St. Antony or a Blaise Pascal, but never an Apollos, never a Philip, never a Barnabas, and never a Paul.

It appears then that Bousset has rightly recognized the divine Jesus, i. e., "the monotheistic cult of the Jesus" (Deissmann), as the energetic element of proto-Christian life, *but he has failed entirely to connect it with a human earthly Jesus,* in fact, he has nearly shown the impossibility of any such connection. The Lord Christ Jesus, Son-of-Man in Heaven, has naught to do with "the historic Jesus," the fictive Carpenter of Nazareth. Above and beyond all question the former is independent of any human earthly life of Jesus, indeed it antedates any such life, and alone is present and effective in the early church; the latter is at best both problematic and functionless—it explains nothing but renders all else unexplainable. Why then retain such an imaginary unconnected with any

other symbol in the equation whose solution it embarrasses, yea, makes impossible?

It is upon this fatally weak spot, this yea-nay in Bousset's theory, that a critic equally acute and friendly (Max Brückner, in *Theol. Rundschau,* May, 1914), in a highly appreciative and deeply sympathetic review has laid firmly the finger of kindness. Speaking of Bousset's correct doctrine that "this belief in the exaltation of Jesus as Son-of-Man was not the consequence but much rather the presupposition of the appearances of Jesus," and of Bousset's attempted psychological explanation — through "the incomparably powerful and indestructible impression, which the personality of Jesus had left in the soul of the disciples and which was mightier than public shame, death, anguish, and overthrow,"—that the disciples "had no other choice" but to transfer the *already made* concept of the Son-of-Man to the Crucified, Brückner declares both wisely and frankly: "I must confess that these psychologic discussions of Bousset's do not satisfy me" (p. 173). After exposing the futility of Bousset's assertions he adds emphatically: "In no case can the screaming dissonance of the crucifixion of Jesus and his exaltation as Son-of-Man in the faith of the *Urgemeinde* be resolved solely by psychologic considerations." Yet *no other considerations has the historicist to urge.*

At this point, then, we must pause. The work of Bousset is no less of prime importance elsewhere and particularly in the treatment of Paul; especially noteworthy is his just judgment (p. 143): "It may be definitely maintained that what we call the moral religious personal character of Jesus had no influence and no significance whatever for the religious feeling of Paul." "The Jesus that Paul knows is the preexistent heavenly Christ," who alone is "the subject to all these predicates," and "not the historic Jesus" (p. 144). All of which is most just and true and shows to what position Bousset has advanced, a tent wherein he takes his noon-day rest, where it is pleasant to stop but impossible to stay. It would be interesting to determine yet more exactly the angle through which this great work marks the rotation of the critical firmament, did not space fail for any such measurement; but no one can lay it down after careful perusal and not exclaim, with or without Galilei, *And yet it moves.*

At the close of the leading article in the *Theologische Rundschau* of October, 1911, our author tempered his hostile criticism of the second edition of *Der vorchristliche Jesus* with these words:

"But these deviations of Smith's researches possess and preserve in their very forcefulness and originality a power of stimulation and of invigoration. They compel us to enter more carefully into difficulties and problems which investigation has hitherto passed by with indifference and without regard, and they help perhaps to bring many a new result of investigation forward to the light." This, our author's present volume, may be taken as a fitting commentary upon his earlier text.

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VEDANTISM, ITS INTRINSIC WORTH AND ITS VAGARIES.

Vedantism, the philosophy of ancient India, which sets forth the end or purpose of the Veda, the religious books of the old Brahman religion, is one of the most interesting and important phases in the history of philosophy. It is a remarkable attempt of ancient Hindu thinkers to reach a finality of thought by an intuitive comprehension of existence. No one who has become accustomed to scientific ways of thinking can approve this system of philosophizing, and least of all can he see a finality in it. To him the solutions offered are merely empty phrases which do not solve the great problems of existence that science of to-day undertakes to fathom by methodical investigation, by logic and rational thought, by experiment and by the systematization of all knowledge into one unified and consistent whole.

A study of the Vedanta is highly to be recommended, for we should understand it and be able to feel its grandeur, its beauty, and the truth it contains. It is necessary to grasp its truth in order to see that its truth is relative; an understanding of the relative character of its truth reveals its insufficiency; and, seeing its insufficiency, one transcends it, satisfied that there is no royal road to philosophic truth, or to a mystic intuitive wisdom such as that promised by Vedantism. A study of such systems as the Vedanta leaves one with a wholesome respect for and satisfaction with the results of scientific method which, though generally slow and tedious, is sound and sure.

The beauty of Vedantism has been felt by our American poet-

philosopher Ralph Waldo Emerson, who sums its truth up in these lines:

"If the red slayer think he slays,
Or if the slain think he is slain,
They know not well the subtle ways
I keep, and pass, and turn again.

"Far or forgot to me is near;
Shadow and sunlight are the same;
The vanished gods to me appear;
And one to me are shame and fame.

"They reckon ill who leave me out;
When me they fly, I am the wings;
I am the doubter and the doubt,
And I the hymn the Brahmin sings.

"The strong gods pine for my abode,
And pine in vain the sacred Seven;
But thou, meek lover of the good!
Find me, and turn thy back on heaven."

All is life, all is aspiration, all is pressing onward to victory; all is God, and we must understand that God is borne on the creative billow of the All, as well as we. Every finite thing passes—as a phase of the Infinity—but the Infinite endures forever. The conception of Vedantism, as presented in Emerson's beautiful lines, is a kind of pantheism, in which God is the All. God reveals himself in hammer and anvil, in action and reaction, in energy of all kinds, in good and evil, in the aspiration of the worm that crawls in the dust, and of the heaven-inspired prophet who longs for the beyond that he beholds in his vision.

The present number of *The Monist* contains two articles which reflect the spirit of Vedantic philosophy; one is "The Conception of Brahma" by Mr. Leo C. Robertson; the other, "The Vedantic Approach to Reality" by S. Radhakrishnan. Both are splendidly written, both breathe the enthusiasm which as a rule thrills the Vedantic thinker; both are more than mere historical reproductions of the old Vedantic theories, for they offer presentations of Vedantic thought in a modernized form and brought up to date by supplying it with the support that comes from Western thought, thus making its theories more acceptable to the generation of to-day.

In reading these articles one learns, if he has not before become acquainted with Indian thought, to appreciate the Vedanta, and one

may even be led to study Vedantism and its ancient classical documents, the Upanishads. In this the reader will be well repaid, and in his task he could find no better hierophant to introduce him into all the many details of this system of thought than Paul Deussen who has translated the Upanishads into German, and is the author of the most exhaustive treatise on the Vedanta.[1] Study Vedantism and you will be glad to become acquainted with this remarkable phase of human thought, but be not disappointed if after all your trouble you find out that all truth is not contained therein.

In his article on the Brahma conception Mr. Robertson sets forth the main doctrines of Vedantism. I will refer here to the tersest gems of thought, which recapitulate in brief the main "truths" of Vedantism.

If an inquirer is met by any object which he does not understand, be it reality as a whole or one of its finite parts, he solves the problem by the dictum *Tat tvam asi,* "That art Thou"; and the fundamental idea of all philosophy is *Brahma-atman-aikyam,* which, freely translated, means: "existence is Brahma, i. e., God; thou art the self; and both, the Brahma and thou the self are one." Thus the riddle is solved, and one can say, *Aham Brahma asmi,* i. e., "I am Brahma." This is the truth, and this exhausts all wisdom; or in other words, "There is but one and that art Thou."

This is the central truth of Vedantism, and Mr. Robertson sums up the whole doctrine as follows:

"The whole of Eastern mysticism, or for that matter of any mysticism, may be summed up in the compound word *brahma-atma-aikyam,* i. e., the unity of the Brahma and the self. The significance of this is that there is only One real being, a Being that is absolutely One, and as the Vedantist goes on to add in his famous formula, *Tat tvam asi,* 'That art Thou.' The self or soul in each of us, this is the Absolute. But there is not a plurality of selves. There is only One, and That art Thou. Thus boldly the Hindu philosopher declares *Aham Brahma asmi,*—'I am Brahma.' Thus does he identify the individual self with the eternal principle of all Being. Or, if one prefers to use the word God, there is naught but God and that art thou. The individual self is not a part of the Absolute nor an emanation from him, but it is absolutely identical with him."

The philosopher tries to understand the Absolute, but his

[1] *Das System des Vedanta.* The work has been translated into English by the Vedantist scholar Mr. Charles Johnston (Open Court Publishing Co.)

labors are in vain. The Vedantist's answer will be again and again, *neti, neti,* "It is not so, it is not so."

Thinkers and philosophers attempt to unify knowledge. They do it in various ways, and build up different systems of monism, materialistic, dynamistic, spiritualistic, pantheistic, and other varieties. Vedantic monism is thus set forth: The world is one because its oneness is my oneness, and I myself am Brahma, the world principle. I am the All. Brahma can be characterized only negatively. *Neti,* "it is not thus." If we want to know more we are told that the Absolute exhausts all. It is the end of all our study, and our knowledge of it must satisfy us. The ultimate result is, "I am I."

We thus sink into an abyss of definitions which have no meaning, but we must not mind, for the All is Brahma, and I am the self; but the self is Brahma, and Brahma is the Absolute. The circle widens not only into the Infinite, but even into the Naught, for we must know that an absolute is really equivalent to Nothing.

These are the doctrines of the Vedanta in a nutshell, and these sentences often intoxicate the philosophically inclined. We bow down in reverence, we rise in glory, sanctified by the thought of our deified nature. We have fathomed the deepest truths and expanded into the all-embracing divinity of the vast Infinite, the Nothing.

Mysticism has its great rewards; it has beauties of its own; it builds up for the mind a heaven of its own. It pleases our mind; it satisfies our intellectual needs; it fills our soul with enthusiasm and with a religious intoxication. Verily, it is grand and magnificent; it fills man with the divine spirit and reveals to him his own godhood. He is no longer a finite creature; he is the Infinite, the Absolute,—God himself. Tell the scientist, the philosopher, whoever is still searching for the truth, that he need no longer vex his soul by searching for it in painstaking investigations, for the truth has been found. Here is the truth in three words, *Brahma-atman-aikyam.*

Will the scientist, or a scientifically trained man, accept this verdict? No, he will not. The scientist's answer is rarely complimentary when the Vedantic gospel is preached to him. I will boldly repeat what different scientists have said and what others will say when they hear such truths as these proclaimed: "Brahma is the atman, and the atman is myself, and I am Brahma." The scientist will not shout hallelujah, or hosanna, but will ask, "What

does this mean?" And after troubling for a while he will probably come to the conclusion: "All these sentences are fine phrases, but they are unmeaning and empty. They do not help, they explain nothing; and if I try to decipher their meaning they prove to be simple nonsense. Tell me what they can mean, and I will try again." To this the Vedantist will say: "How narrow are these modern scientists; their minds are closed to the deepest truths." That is and will remain the end of the controversy, and we must recognize that there is an unfathomable abyss between the two parties.

We might close here, but we must not withhold from our readers the fact that the Vedantists are not only misunderstood to-day, but they also met with severe opposition in ancient India, and their great adversary was Gautama whom still to-day hundreds of millions of human beings worship as the Buddha, the Enlightened One.

In Buddha's time Brahmanism was the religion of India, and Brahmanism preached a belief in Brahma, the existence of the self or the atman, and salvation from evil by prayer, sacrifice and other religious ceremonies. Buddha opposed the main doctrines of Brahmanism, and declared that prayers, sacrifices and ceremonies were of no avail and that man can find salvation only by purifying his heart, by avoiding evil, and by doing good wherever he could. His doctrine is summed up in this quatrain, translated from the Dhammapada:

> "Cease evil and do good,
> And let thy heart be pure.
> This is the truth of Buddhahood,
> Which will for aye endure."

But Buddha opposed also another important doctrine of Brahmanism. He rejected the theory of an atman and preached the doctrine of the anatman, the theory that there is no self, or rather that what we call self or atman is a combination of several qualities but not an existence in itself.

It is a coincidence that in his doctrine of the anatman Buddha anticipated modern psychology, with its scientific conception of the soul, and if I recommend the study of Vedantism I will not omit to advise my readers not to overlook Buddhism. They will have to choose between the two; *tertium non datur.*

Mr. S. Radhakrishnan presents us right at the outset with about half a dozen different definitions of philosophy. We do not

know which of them he accepts as its central and most important feature, but it does not matter very much which of them he would select; for none of them seems to him sufficient; all seem needed to bring out the complete significance of philosophy.

Philosophy is, to Mr. Radhakrishnan, "the attempt to think out the presuppositions of experience, to grasp, by means of reason, life or reality as a whole." I suppose the Vedantist, with his modern education, here has in mind Kant's transcendentalism, which systematizes the presuppositions of experience, such as transcendental logic, transcendental esthetics, etc.

The presupposition of experience is an important domain of science. It comprises what Kant calls *a priori* thought, and consists of the purely formal sciences, logic, arithmetic, geometry, and the purely natural science (i. e., the doctrine of causation). But this group does not constitute philosophy. At best it is but an introduction to philosophy. Philosophy is more. Philosophy is the conception of the world as a whole. And this broader conception of philosophy would presuppose a systematization of the results of all the sciences into what Comte calls a hierarchy of the sciences. A systematic description of the whole is different from Kantian transcendentalism, and it might be a special task of the Vedantist to try to reconcile the two. A reconciliation is not impossible, but certainly we should have to overcome some difficulties.

The Vedantist however is confronted with additional problems. According to a third definition philosophy "has to find out an all-comprehensive and universal concept which itself requires no explanation while it explains everything else." Many philosophers have tried to find a universal concept, but all of them have failed. Materialists have found this universal concept in matter, but they have not succeeded in deriving everything else from matter. How, for instance, can we derive from matter the truth of geometrical theories? Logic is not explicable from matter, nor can its principles be derived from material phenomena. Further, life cannot be explained from purely mechanical principles, and still less feeling and consciousness. The truth is that in this world there are several distinct universal concepts. Life cannot be derived from dead matter, nor inversely can the meaning of matter be derived from the notion of life, motion, or energy. Even the two ideas, matter and energy, are absolutely distinct and different. Energy is change of place, and all we can do is to declare that it is closely connected

tific explanations. I am too narrow, too prosaic, too unmetaphysical. Such is the disposition of my mind, and I cannot help it."

A mutual friend from the distant East had in the meantime approached, and the professor turned to him, saying: "Excuse me for my inability to grasp your truth. Pity me if you will, and pray for my soul, but I fear I am a hopeless case. The facts as I know them are rigid and horribly obstinate things, while your Vedantic thoughts are beautiful, artistic and charming, but vague, inexact and unscientific; but they are fragile and fall to pieces at a touch. Science has its faults; it is narrow and one-sided. I am a scientist; bear with me."

Just one more point,—in reference to the Vedantic term atman, which means "self" and denotes the "soul." The idea of the soul as a metaphysical entity is probably a very old conception, and must have existed in the days of Gautama Buddha, the founder of the new religion which gradually spread over the valley of the Ganges and then over all of Asia, but was finally exterminated in its original home, India. Then Brahmanism was reintroduced and insisted most vigorously on the very doctrine that had been combatted by Buddhism—the doctrine of the atman, the existence of the soul as an independent self.

Buddhism is very modern in its philosophy, and emphasizes the positive and scientific aspect in religion and philosophy. Buddha claims that the soul is not an intrinsic unit, but a cooperation of psychic activities; and at the time when Buddhism was proscribed and the older Brahmanism restored, the philosopher of this reformation was Shankaracharya, the systematizer and formulator of this theory of the atman.

It would take a long article merely to discuss the meaning of the atman, and I will not enter here into details as I have discussed the subject repeatedly.[2] I will simply say that the term atman in Vedantism is the hypostatization of a general concept into a concrete actuality, a procedure which is apt to produce the metaphysical notions in the domain of philosophy. So we shall have to deal with it in the same way, and it will come to pass that the scientific psychologist will be regarded as a nihilist by metaphysical thinkers, just as Buddha's psychology is denounced as "a psychology without a soul."

The belief in these metaphysical entities has become so impor-

[2] *Buddhism and Its Christian Critics*, pp. 87 ff. *Open Court*, X, 4851.

tant to the Vedantist that to him the simpler and purely scientific view seems irreligious and infidel. It was Buddha who proved to the world that a religion, yea, a very devout and stern religion, can be built up upon the most radical foundation. And why? Because the real self is as important as the shadowy metaphysical self, and if the atman is treated as an eternal unit, as in Vedantism, it may serve as a symbol of the character of a man, as his mind, his spirit, his heart, or his soul, and in this sense Buddha teaches his disciples to rid themselves of their impurities as a silversmith blows off the impurities from the silver when preparing it for his furnace (Dhammapada, 239).

In the original, Buddha here uses the very term atman, not of the metaphysical soul-self, but of the actual self, the personality of his disciple. In practical questions, both the believer in a metaphysical soul and the philosopher of the anatman lay down the same moral maxims, but in theoretical explanations, we have the two views in contradictory opposition, the rigorously scientific view and the artistic vision of an attractive but hazy mysticism.

EDITOR.

WHAT IS INTUITION?

What *is* intuition?

The dictionaries define the word as follows:

a. "A looking upon; a seeing either with the physical eye, or with the 'eye of the mind.'"

b. "Direct or immediate knowing; truths known by intuition are the original premises from which all others are inferred; intellectual intuition is applied to mystical vision; innate conceptions of right and wrong."

c. "Any object or truth discerned by direct cognition; a truth that cannot be acquired by, but is assumed in, experience."

d. "Pure untaught knowledge."

e. "The term intuition will be taken as signifying a cognition not determined by a previous cognition of the same object, and, therefore, so determined by something out of the consciousness."

f. "Comprehension of ideas independently of ratiocination; innate or instinctive knowledge."

But these definitions are not, after all, very illuminating; they give no clear idea as to what intuition *is*. They create a certain inference: we seem to see in intuition a remote influence through which the attitude or the conduct of the individual is influenced. This influence appears to be highly subtle, having an apparent origin either in a higher plane of the human mind or in a plane higher than that of the human mind; it appears to be either ultrahuman or suprahuman. And our association with the word, and our experience with the phenomena of intuition have been such that we are ready to accept intuition at just this vague and mysterious valuation. But now are we right in so accepting it? My answer to that question is, No.

I look upon intuition as nothing more than a product of normal brain activity; there is nothing ultrahuman or suprahuman about it. I take it to be just reflex cerebration.

The grey cells of the human brain possess a function which is peculiar to those cells; and this function, which no other cell possesses, we name self-consciousness. But, in addition to this, they possess another form of consciousness, namely, reflex. In self-consciousness the action of the cells is directed from within, whether the action results in thought or in the direction of a muscle. In reflex consciousness action can be aroused only through external stimuli. Reflex consciousness is the primal form of consciousness, for the primitive brain, in the lower orders of evolution, was made up only of reflex centers. As these centers evolved, as the brain increased in size and the area of the cortex increased, the new self-consciousness became associated with the old reflex consciousness. As the animal developed he became more and more aware of what he was doing; his higher centers took command. But at first these higher centers were little better than the others. They had developed through the lower, and, because of this, their mode of action was the same; they were only reflex centers, even though they were thought centers. The animal began to think, but his train of thought could be started only through an outside circumstance. His thought lacked value for the reason that he was, as yet, weak in the two essentials to good thought: memory and experience. As time went on, continued use of the new centers developed their function. Experience became wider, and memory grew stronger; and, as

memory developed, self-consciousness came into being. Self-consciousness was now supreme, for it was through this only that thought could be directed and regulated. Without self-consciousness thought would be only reflex thought, automatic thought, thought not controlled by the indivdual.

Of the two forms reflex thought is the older; it is, in fact, the first form of thought, basic thought, and it will, therefore, always exert its influence upon the cerebral cells. And even though self-conscious thought has become the dominant influence in the cerebral cortex, reflex thought still has a place there. It has existed as long as the brain has existed, for perhaps a million years, and its influence is going to persist for an indefinite period, perhaps as long as the brain of man endures.

We find then that the higher centers have reflex action just as the pure reflex centers have; the one is as easily excited to action by an external stimulus as is the other. But there is this difference between the two: Whereas pure reflex action is non-conscious action, this thought-reflex may be either non-conscious or conscious; the individual may or may not know what his thought cells are doing. These cells were created to act in a certain direction, and they always act in that direction, whether controlled or uncontrolled. *They do not need the direction of the individual in order to act, although they act better when so directed.* Thought can go on without the participation of the individual. The cells having been developed for the purpose of thought, and having performed that function for countless centuries, cannot avoid the thought-reaction when excited by the proper stimulus. The action has become a "habit-action," and through force of habit the cells think, even before the individual is aware. The fact that thought goes on in our dreams will prove this.

Intuition, then, is reflex thought; it is habit-action of the thought-cells, non-conscious action. The period of this non-conscious action is usually very short: the cells perform this non-conscious action, and then the individual becomes aware of the action. But that which he perceives is, not the action itself, but the result of the action. He perceives this result, and, not knowing that his own cells have evolved that result, he calls it "intuition," that is, a supernatural admonition. It does no harm to call this thing intuition; but it is wrong to give it that exalted value. It is merely reflex thought, without the value even of self-conscious thought.

The value of thought depends upon two things: the extent of the individual experience, and the degree of the individual self-consciousness. If a person has had a wide experience of a subject under discussion, and if he has been trained to think, if he knows how to use his experience, then his thought is going to be of value. His thought *must* be guided by his self-consciousness. Now reflex thought is not so guided. Cell action, whether conscious or non-conscious, is determined by cell experience; but if the action is outside of the individual consciousness it lacks the essential attribute of real thought. At its best it is nothing more than half-thought. It is true that this attempt at thought may happen to move in the right direction, but it is only chance if it does; it may as easily move in the wrong direction. If it goes right we call it intuition; if it does not go right we say that we guessed wrong. And so, after all, in what way is intuition superior to a mere guess?

In conclusion let us again go over the definitions at the head of this article. It may be that now they will appear to us in a different light.

Definition *a*. "A looking upon; a seeing either with the physical eye, or with the 'eye of the mind.'"

This comes the nearest of any of the definitions to telling us what intuition is, but it fails to completely enlighten us. Looking upon an object with the "physical eye" may bring intuition into action, but what is the action? So, also, may looking upon an object with the "eye of the mind," but, again, what is the action? We find that what these acts of looking do is to excite thought about the object looked at. If the object is something new to the observer the individual stores up in his memory the new image-impulses which come to him from it; if it is an object with which he already is familiar it recalls to his consciousness the images which former "lookings" have stored there. If now, while we look upon the object, we fancy that we have some subtle knowledge of it, we call that intuition, and we imagine vain things about the high origin of this knowledge. But if we analyze the matter, if we look into our thought-cells, do we find anything in those cells behind those thought-images? Is there anything in the thought-cells that is above thought? The answer to that question will determine the status of intuition.

Definition *b*. "Direct or immediate knowing; truths known by intuition are the original premises from which all others are inferred; intellectual intuition is applied to mystical vision; innate conceptions of right and wrong."

What is meant by the above expressions? What is "direct knowing"? Where are the "original premises"? What is "mystical vision"? What are "innate conceptions"? Are these anything more, in our present light, than figures of speech? The words "direct," "original," "mystical," "innate," have no right to be used here. The use of the words merely elevates the subject to the realm of the supernatural, where it has no place. Being just a process of human thought, it cannot go above thought.

Definition *c*. "Any object or truth discerned by direct cognition; a truth that cannot be acquired by, but is assumed in, experience."

Here we find the word "direct" again, and we voice the same objection to it. There is no such thing as "direct cognition" as applied to the human brain; also, there can be no such thing in the cerebral cells as a truth *not* acquired by experience.

Definition *d*. "Pure untaught knowledge."

I take this to mean knowledge not acquired through experience. There can be no such knowledge in the human brain. If such knowledge were possible some of us would never need to study; but even the genius has to do that.

Definition *e*. "The term intuition will be taken as signifying a cognition not determined by a previous cognition of the same object, and therefore so determined by something out of the consciousness."

The process of intuition *does* take place "out of the consciousness"; but *there must be* a "previous cognition," else there can be no intuition. Intuition, like thought, depends upon experience, upon knowledge.

Definition *f*. "Comprehension of ideas independently of ratiocination; innate or instinctive knowledge."

Both of these may be denied. There can be neither "comprehension of ideas independently of ratiocination," nor "innate or

ideals such as the dreams of a prophet, be he genuine or a fanatic,—or as poetic conceptions expressed in some visualized or visible form.

Poetical dreams of this kind come to the real poet not by the slow and elaborate process of argument but by a prophetic insight,—by a sudden enlightenment comparable to a flash of divine inspiration. This is intuition in the mystical sense.

We need not here enter into details as to the psychology of mystical intuition, its natural origin and development in the realm of the subconscious, and its sudden and unaccountable appearance in consciousness in such a manner as to lend itself readily to a mystical interpretation. Suffice it to say that the uncritical observer receives the impression that even in his own visionary experiences he is dealing with divine inspiration. Intuitions are described as coming to the poet by revelations from on high, and therefore he claims that he does not shape his thoughts himself but discovers them, the subconscious process remaining hidden. He is conscious only of the result which is suddenly presented; the vision is shown to him as if it had existed and is seen only by him because he is a favorite of the Deity, of the muse, or whatever the mystical source and power may be called.

Kant's term *Anschauung* is very different. It does not contain the slightest element of mystic thought. It has reference to the sensation of sight and may frequently be translated by the word "sensation" itself. The difficulty of translating the German word *Anschauung* consists in the fact that there is no English word of Saxon derivation expressing the meaning of that which has become an object of sight, and just as the word *Anschauung* is indigenous German, so the English language should have an indigenous Saxon term to express the meaning of the word *Anschauung*. It is a peculiarity of English words derived from the Latin that they express abstractions. Thus the Latin translation *intuitio* implies the idea of an abstract designation, while a Saxon word composed of purely Saxon elements with the same meaning as *intuitio* would naturally refer to a concrete process of well-known and daily experience. It is for this reason that some time ago, while discussing the difficulties of Kantian philosophy, I proposed the adoption of a purely Saxon word "atsight" to fill this gap in the English. The word "atsight" denotes that which is at sight, so that it can be seen and is actually beheld. The difficulty of the term consists in its newness, but it is easily understood by its etymology and is justified

by analogy. As the eye pictures what is presented to it by being at sight, so the process of looking into the nature of things is called "insight." Thus the difficulty due to the newness of the word can easily be overcome. The Latin word "intuition" exactly translates this new word "atsight," but we must beware of the mystical meaning of it, and, when reading Kant, we must remember that Kant's term *Anschauung* excludes the mystical from its meaning and that this difficulty is presented only in translations.

A new difficulty presents itself when we read Croce's expositions. It seems to me that Croce uses the term "intuition" in a third sense which has an element of each meaning. Unless I am greatly mistaken the visionary element is not absent, and the intuition of the poet is in so far added as Croce distinguishes his term "intuition" from both concept and sensation. Sensation is simply the crude material received by the senses, while intuition embraces what Croce calls "expression," which means that it is worked out into a concrete vision poetically presented, not as a mere definition of an idea but as an artistic picture in all details and concretely individualized.

Whether this view is correct ought to be established by a critical student of Croce's philosophy, or, better still, perhaps Professor Croce himself will tell us whether we have rightly understood his theory.

As to the essential significance of his esthetics we are glad to say that we agree with him thoroughly, although we approach the problem from a slightly different angle. Whatever may be the artist's definition of beauty is a matter of secondary importance, but it is essential to know what art is, how it originates, why it exists and what is its purpose.

Art has been defined as a presentation of beauty, but how often does art present the ugly, the terrible, the dastardly. Tragedy is described by Aristotle as the highest product of art, and it is a struggle between the good and the evil, in which the good suffers and succumbs. Art has been characterized as an imitation of nature; but music is not an imitation of bird songs, otherwise one of Beethoven's sonatas would be a gross aberration from the art ideal. Music builds up a world with its own laws in the realm of tones. It is an original creation at best parallel to the actual world in general, but not an imitation of nature. One feature, however, is noticeable in all arts. It is this, that art presents the world-conception of the artist in concrete definite instances. The artist imitates

nature in the sense that he builds up a world and delineates it before our eyes. The tragedian pictures life as a struggle and points out how a good cause may triumph while its hero sacrifices himself, and the landscape painter portrays human sentiments, or as the Germans say *Stimmung*, in the shape of clouds and trees and atmosphere. In the creations of the artist the chief thing is the spirit or mood which dominates them. Art may describe something beautiful or something ugly, something real and natural or something non-existent, a world of laws extending through infinite space —it is always a creation, always the production of a world, always a description of life and the laws of life.

EDITOR.

FOUR-PLY PANDIAGONAL ASSOCIATED MAGIC SQUARES.

Mr. Frederic A. Woodruff has sent us three original magic squares, one each of orders 8, 12 and 16. The two smaller squares

1	32	34	63	37	60	6	27
48	49	15	18	12	21	43	54
19	14	52	45	55	42	24	9
62	35	29	4	26	7	57	40
25	8	58	39	61	36	30	3
56	41	23	10	20	13	51	46
11	22	44	53	47	50	16	17
38	59	5	28	2	31	33	64

Fig. 1.

1	140	81	36	77	100	73	104	9	108	113	28
143	6	63	110	67	46	71	42	135	38	31	118
25	116	105	12	101	76	97	80	33	84	137	4
119	30	39	134	43	70	47	66	111	62	7	142
16	125	96	21	92	85	88	89	24	93	128	13
131	18	51	122	55	58	59	54	123	50	19	130
15	126	95	22	91	86	87	90	23	94	127	14
132	17	52	121	56	57	60	53	124	49	20	129
3	138	83	34	79	98	75	102	11	106	115	26
141	8	61	112	65	48	69	44	133	40	29	120
27	114	107	10	103	74	99	78	35	82	139	2
117	32	37	136	41	72	45	68	109	64	5	144

Fig. 2.

are four-ply, pandiagonal and associated, and the one of order 16, besides having these features, is also Knight-Nasik.

Mr. W. S. Andrews writes us that these squares are very interesting mathematical curios inasmuch as they probably present

the best combinations of ornate features that are possible in their respective orders. Mr. Woodruff has also devised ingenious mathe-

1	224	49	240	145	80	161	128	113	176	65	160	225	64	209	16
254	35	206	19	110	179	94	131	142	83	190	99	30	195	46	243
4	221	52	237	148	77	164	125	116	173	68	157	228	61	212	13
255	34	207	18	111	178	95	130	143	82	191	98	31	194	47	242
10	215	58	231	154	71	170	119	122	167	74	151	234	55	218	7
245	44	197	28	101	188	85	140	133	92	181	108	21	204	37	252
11	214	59	230	155	70	171	118	123	166	75	150	235	54	219	6
248	41	200	25	104	185	88	137	136	89	184	105	24	201	40	249
8	217	56	233	152	73	168	121	120	169	72	153	232	57	216	9
251	38	203	22	107	182	91	134	139	86	187	102	27	198	43	246
5	220	53	236	149	76	165	124	117	172	69	156	229	60	213	12
250	39	202	23	106	183	90	135	138	87	186	103	26	199	42	247
15	210	63	226	159	66	175	114	127	162	79	146	239	50	223	2
244	45	196	29	100	189	84	141	132	93	180	109	20	205	36	253
14	211	62	227	158	67	174	115	126	163	78	147	238	51	222	3
241	48	193	32	97	192	81	144	129	96	177	112	17	208	33	256

Fig. 3.

matical tables by which squares similar to those here shown can be easily constructed in great variety.

The diagrams of Mr. Woodruff's squares were kindly drawn by Mr. H. Sayles. EDITOR.

CURRENT PERIODICALS.

The number of the *Revue de métaphysique et de morale* which should have appeared in November, 1914, to complete the twenty-second volume, appeared in November, 1915. Victor Basch writes on the classical philosophy and literature of Germany, and pan-German doctrines; Louis Weber writes on affective memory, with reference to the work of Ribot and others; and Pierre Boutroux writes on the historical significance of Descartes's *Géométrie*. This last article does not seem to add much to what we know. There is a critical study by Harald Höffding of the book of 1912 by

the eminent sociologist E. Durkheim on the elementary forms of the religious life. Th. Ruyssen, in an article under the heading "Practical Questions," discusses force and law. A supplement contains a list of the courses of lectures on philosophical subjects given at the universities in France and Switzerland, and reviews of books and periodicals.

* * *

In *Scientia* for December, 1915, Gino Loria gives a rather slight sketch of the ideas of the ancient Greek mathematicians on the infinite and infinitesimals. Charles Fabry continues from the preceding number of *Scientia* his article on luminous atoms and their motions; this part of his article is on the constitution of the luminous atom and he remarks that the notion of atom loses more and more its etymological meaning. W. H. Bragg describes his new methods of studying crystalline structure by the X-ray spectrometer, which open up an entirely new method of describing the characteristic features of crystals. Ramsay Muir discusses the problems of future peace in a reprint of the preface to the English translation of Rignano's article in the number of *Scientia* for June and July, 1915. Prospero Fedozzi writes on the teaching of the war with regard to the treatment of foreigners. E. S. Russell discusses recent books by Bateson and Ruggles Gates in a general review on the problem of species and their origin. There are also reviews of books and periodicals, a chronicle of events, and French translations of the Italian and English articles.

In the number of *Scientia* for January, 1916, Gino Loria continues his article on the infinite and infinitesimal. After shortly sketching the way in which science came out again from the darkness of the Middle Ages, the author gives a very able sketch of the progress made by mathematicians, from this time to the end of the seventeenth century, in the conceptions which finally led to the infinitesimal calculus. Percival Lowell gives a short but highly interesting paper on modern work relating to the atmosphere of Mars. Hugo de Vries writes on the evolution of organized beings in a discontinuous manner, which is proved by the work of Nilsson. There are two articles on questions raised by the war: one is by Augusto Graziani on the future economical consequences of the war, and the other is by André Weiss on past and future international law. There are also the usual reviews of books and period-

icals, a chronicle, and French translations of articles in Italian and English.

* * *

The Bulletin of the American Mathematical Society for December, 1915, contains articles on absolutely continuous functions (M. B. Porter), on the representation of numbers in a certain form (R. D. Carmichael), on the linear continuum (Robert L. Moore), and a problem in the kinematics of a rigid body (Peter Field). R. C. Archibald gives a very interesting and apparently complete list of memoirs on Henri Poincaré that have appeared of late years. The beginning of this article is a review of some books on Poincaré's life and work, and there are several other reviews in this number, marked, as reviews in the *Bulletin* always are, by great learning and critical ability. There are also some notes and a classified list of new publications on pure and applied mathematics. Moore's article just mentioned is of particular interest to the readers of *The Monist*, as it is concerned with the logical question of a set of axioms for geometry. This subject has always appealed strongly to American mathematicians, and their work stands by the side of the splendid work in this direction of Pasch, Peano and Pieri, and is markedly superior to the later work initiated by Hilbert, which has attained such fame. The sixteenth volume (1915) of the *Transactions of the American Mathematical Society* contains renewed proof of this: Robert L. Moore of Philadelphia writes on a set of postulates which suffice to define a number plane, and Meyer G. Gaba of Pittsburg writes on a set of postulates for general projective geometry. Besides this, there are two exceedingly interesting papers in French: one by Maurice Fréchet on bilinear "fonctionelles," and a long and important work by Charles de la Vallée Poussin, now of Cambridge, Mass., on Lebesgne's integral.

* * *

In *Science Progress* for January, 1915, S. C. Bradford discusses (i) color and chemical structure, (ii) the Liesegang phenomenon in gelatin and allied substances; Dr. David Ellis writes a beautifully illustrated paper on the iron-bacteria; Sir Ronald Ross continues his important mathematical researches on the solution of equations by operative division; and T. A. Mason concludes his deeply interesting study on the influence of research on the development of the coal-

tar industry. There are also essay-reviews and a large number of other reviews of books, full accounts of recent advances in all branches of science, and notes and correspondence. Φ

KARL EUGEN NEUMANN.

Word has been received from Vienna announcing the death, on October 18 last, of Dr. Karl Eugen Neumann, an oriental scholar who opened to modern readers a larger part of the Buddhist Pāli canon previously untranslated than probably any other man, with one or two exceptions. Having been born October 16, 1865, he had passed the half-century mark by just two days when he died. The cause of his death has not been learned, but his age rebuts the presumption, which otherwise would be strong, that he was killed in the war. In spite of the reports of almost universal draft which come from his city, as from Teutonic countries generally, it is hardly likely that a man of learning so near the age limit would have been taken. This war has indeed wrought great havoc in scholarship. At its beginning some half dozen German philologists were editing manuscripts for the (English) Pāli Text Society, which has been left wondering whether they are still alive. Neumann was not one of these, apparently; his work consisted in converting the ancient into a modern tongue. His fame was limited by the small circulation which oriental literature, when presented in a full and faithful form, almost invariably receives; and the fact that he wrote in German barred him from this country, where an intelligent knowledge of that language is not common, in spite of the many who are acquainted with it colloquially or superficially.

Dr. Neumann's work of greatest interest was the publication, in 1899, through Ernst Hofmann & Co., Berlin, of his *Lieder der Mönche und Nonnen Gotamo Buddho's,* being the first translation of the Therā-therī-gāthā. This is perhaps rightly to be judged the most important collection of verse in the Buddhist canon, not even excepting the Dhammapada. As an evidence of personal religious experience, it is one of the most significant books in all literature and is likely to become celebrated when, twenty-five or fifty years hence, writers on religious psychology discover it. Dr. Neumann treated these hymns in a manner which has been very unusual in dealing with Buddhist poetry—he rendered them rhythmically and gracefully. His stanzas—iambic tetrameter blank verse quatrains for the most part—abound in happy phrases, are distinguished by a peculiar dignity, and possess a melancholy charm of sound which goes far toward suggesting the feeling appropriate to them. No translation of these gāthās appeared in English until ten years later, when Mrs. Rhys Davids brought out the *Psalms of the Early Buddhists,* the Sisters in 1909 and the Brethren in 1913.

A work of greater magnitude by Neumann was his translation—much of it for the first time—of the Majjhima and Dīgha Nikāyas, vast collections of homilies or dialogues attributed to Buddha, and of such early date that they may be considered as containing much that he really uttered. These books have long been recognized as embodying the doctrinal substance of the Pāli canon, and many selections of them had before been made, but it remained for

Neumann to perform the heroic task of coping with them in bulk. Their publication (here assumed to be complete) under the title *Die Reden Gotamo Buddho's* extended over the years 1896-1912. A minor work, printed in 1892, the *Buddhistische Anthologie,* had contained some extracts from the two Nikāyas mentioned, as also from the Anguttara Sangyuttaka. He also published (1905) a version of the Sutta Nipāta, which had already been done into English.

As a philologist Neumann had the courage to defy convention by following the actual Pāli nominatives of nouns, thus writing "Buddho" and "Gotamo" instead of "Buddha" and "Gotama."

For a well-rounded sketch of his life, data are not at hand as this is written. His birthplace, it may be added, was Vienna; his education was obtained at schools there and in Leipsic, at a Higher Gymnasium in Prague and at the Universities of Berlin and Halle. It is fitting that all students of Buddhism should acknowledge his great and splendid achievements and remember him as a scholar of distinction in his chosen field. Particularly is this recognition due now in America, where there exists a conspiracy of spite against all things German, and where a great popular lecturer who feels himself called to the moral instruction of mankind abandons a course in German literature from malice. One who does not claim a drop of German or Austrian blood is glad to offer this tribute to Karl Neumann. As an exponent of a religion incommensurable with violence, it should be gratifying to feel that in the midst of war he probably died a peaceful death and that thus the consistency of his life-long devotion was not shattered. EDWARD P. BUFFET.

NOTES.

Prof. A. H. Lloyd of Ann Arbor, Michigan, has published in *The American Journal of Theology* of January, 1916, an essay on "Incarnation," which treats the subject in three parts: first, A Modern Superstition; second, What Ideals Are Made of; and third, Some Practical Values of Mystery.

Our author concludes his essay as follows: "I set out to speak of the values of mystery. There were three to which I wished to call attention. Mystery was the background of real opportunity. Mystery brought to life a saving humor. And mystery could make its object real only by making it an object of will. A world of untold opportunity, of nothing less than the opportunity of incarnation, realizing the spiritual in the natural, finding the ideal in the actual, stands before the will of the present day."

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THE MONIST

THE HISTORY OF SCIENCE.

INTRODUCTION.

[Dr. George Sarton is a Belgian scholar who has done much to promote the idea of a "History of Science" (as opposed to the history of any particular science, or to the sum of such particular histories.) He advocates a synthetic study that necessitates the collaboration of the scientist, the philosopher and the historian.

In 1913, Dr. George Sarton founded *Isis*, an international quarterly devoted to the history and to the organization of science, printed and published in Belgium. He himself lived a very quiet and retired life with his wife and daughter in his country home of Wondelgem, near Ghent—devoting all his time and a great deal of money to his historical studies. When the German invasion broke over Belgium, their income being entirely cut off, they had to leave their home; and after having buried all manuscripts in their garden, they went in a peasant cart to Holland, thence to England, and lastly came to this country. Dr. Sarton's library—one of the most complete on the subject he is studying—had to be abandoned: we sincerely hope that it will be saved and that Dr. Sarton will recover it after the war. He lectured in 1915 on the history of science at the summer school of the University of Illinois, at the George Washington University of Washington, and at Clark University. He has now been appointed lecturer at Harvard.

Dr. Sarton will resume the publication of *Isis* as soon as circumstances permit.—EDITOR.]

THIS essay is to explain the meaning of the history of science, to determine its limits and to show how it should be studied.

The history of science is the study of the development of science—just as one studies the development of a plant or an animal—from its very birth. We try to see it grow and unfold itself under many diverse conditions. And it is not enough—as we shall see further on—to study sep-

arately the development of each science; one has to study the development of all the sciences together. Besides, it is impossible to separate them satisfactorily one from the other; they grow together and mingle continually in innumerable ways.

There has been much research concerning the history of some particular sciences, and there are, for instance, excellent textbooks on the history of mathematics and of medicine, but there does not exist at the present day even a tolerably good history of science. The reader very likely knows the *History of the Inductive Sciences* by William Whewell, published in 1837. It was certainly a valuable book seventy years ago, but is now antiquated, and any one who does not know the history of science will do better not to use it at all. The best book that we have now at our disposal is that of Friedrich Dannemann,[1] but it is very elementary and can only be considered as a first and rough approximation. A bulky work published by Henry Smith Williams seems to be very popular in this country; at least, I have found copies of it in all the libraries where I have been. They are generally placed in the reference room where they are likely to be very often consulted. Owing to this, I feel obliged to say that these books are nothing but newspaper work, and quite unreliable.

While numberless books, many of them excellent, are published every year on the history of literature, of art, of religions, how is it that there is not yet a single history of science that can be compared with the best of them? When so many institutions, libraries, lectureships have been dedicated to the history of everything, how is it that the history of science has been so much neglected? The history of everything has been studied, except of that which

[1] Friedrich Dannemann, *Die Naturwissenschaften in ihrer Entwicklung und in ihrem Zusammenhange*. 4 vols., 1910-1913. I have analyzed this work at some length in *Isis*, II, pp. 218-222.

is the most distinctive feature of our civilization. How is that?

The most obvious, if not the best reasons, are the following. The people who have no knowledge of science, or but slight, are afraid of it. They are not inclined to read a book dealing with the history of science, because they think that they are not equal to appreciating it. Now this is a mistake: every intelligent man or woman can understand the development of science, at least if it be properly presented and taken from the beginning. More than that, I am convinced that the historical method is the best to convey scientific facts and ideas to unprepared minds and to make them thoroughly understandable,—at least that is so in the case of grown-up people. On the other hand, those who know science—or who are supposed to know it because they have made a special study in some narrow field—are often given to viewing history with contempt. They think that it is hopelessly inaccurate and, according to their own definition of science, unscientific. This is another mistake, which, however, it would take too long to completely refute. Suffice it to say that historical studies, like all other studies, are approximate; the approximation obtained by historians may be looser, but the studies are none the less scientific for that. It is not so much its degree of approximation, as a definite knowledge of this degree that gives to a study its scientific character.

At any rate, these reasons are only the most superficial ones. To set forth the others, I am obliged to make a short philosophical digression.

SCIENCE AND PHILOSOPHY.

Indeed, to make the real significance of our studies clear, it is necessary to impress the reader with a sense of the intellectual needs they must satisfy.

New scientific facts are discovered every day all over

the world and they continually make it necessary to revise our theories or to invent new ones. At the same time, science as a whole becomes more complete and deeper. Since the last century, its complexity has been developed to such a degree that now one of the first conditions of really original work is that it should be sufficiently specialized. The necessity of separating the difficulties in order the better to solve them, has made it more and more necessary to divide scientific work, and this division of labor seems to have reached a climax. That this tendency, which we may call the analytical tendency, has been extremely useful, the whole fabric of modern science is there to testify. However, its exclusive predominance is not without danger. This was not palpable in the beginning, but we see it clearly now. Indeed, the object of science is not to discover insulated facts, but to coordinate and to explain them one by the other. By dint of specialization, science would run the risk of missing its very aim; the quantity of scientific knowledge would increase, but it would be all in vain, the scientific spirit would be impoverished.

Besides, excessive analytical tendencies, without any counterpoise, would bring about another and a still graver danger: not only science would be menaced by disintegration, but our social life itself. Instead of bringing their fellow men together by giving them some common points of view, the scientists would finally be unable to understand one another.

This essential rhythm of our mind that makes us feel by turns the need of analysis and the need of synthesis, we find also in the changing idea that men have of the relations between science and philosophy. Indeed, there corresponds to it a similar rhythm which by turns brings together or drives asunder the scientist and the philosopher. A comparative study of the history of science and

of the history of philosophy would give us many opportunities to verify this.

The scientists of genius—I so call scientists whose discoveries revolutionize all accepted ideas and who originate studies of a radically new kind—have always exerted a considerable influence upon the evolution of philosophy. On the other hand, their own minds must have been of a very synthetical nature, and they have certainly borrowed much in a more or less conscious way from the philosophical store to formulate their revolutionary ideas. Think of Galilei, of Kepler, of Newton, of Darwin. Their work and influence cannot be understood, unless one takes into account these continuous interchanges between science and philosophy. They have drawn the desire of creating a new synthesis from the ideology of their time; and on the other hand, it is because their discoveries have deeply transformed this ideology that their influence has extended far beyond the scientific field where it originated.

In the same way the great philosophers—those who have really renewed the thought of their age—have also considerably influenced the progress of science. They were not themselves creative scientists, but at least they possessed all the scientific knowledge available to them. Think of Plato, Aristotle, Descartes, Leibniz, Kant. Here again, it is indispensable to conceive a double stream of ideas between science and philosophy. It is in the scientific domain that they have found at the same time the intuition of and the materials necessary to a new system; and this system in its turn, has renovated the philosophical atmosphere in which science was to pursue its development.

Therefore, those who study the history of philosophy ought to know the history of science. This is for the philosopher a heavy task, but I do not see how he can possibly escape from it. If one confines oneself to the study of, let us say, Descartes's philosophy, regardless of its conse-

quences in the field of mathematics, mechanics, astronomy, physics, medicine, botany, it stands to reason that it is impossible to give a complete or even a fair idea of his genius. Moreover, it is necessary to study the influence exerted by the Cartesian philosophy over the whole scientific thought of the seventeenth and eighteenth centuries, and even over our own science, and it is only in this way that Descartes's personality appears in its true light.

Everybody remembers those great epochs of synthesis of which Greek antiquity has given us some glorious examples, and nearer to us, the Renaissance and Cartesianism. On the contrary, during the nineteenth century, the analytical tendencies have been predominant. Synthetic construction sank into disrepute, partly as a result of the immense success of the inductive sciences, partly because most people were sick of the loose literature of the metaphysicians who came after Kant.

Whatever the case may be, a philosophical reaction was unavoidable, and this reaction still holds good, our own studies being only one aspect of it among many others. This reaction dates from the beginning of our century; it was in a great measure caused by the resounding discoveries of the last twenty-five years. First of all, the progress of physics has involved a conflict—that seemed first to be inextricable—between the classical mechanics of Galilei, Huygens and Newton, and the electromagnetical theories of Maxwell, Hertz and Lorentz, and so has brought into question the fundamental principles of natural philosophy. At the same time, the discovery of new elements having paradoxical properties, the study of new radiations, of the Brownian movement, rekindled all the controversies relating to the atomic and energetic theories and obliged the scientists to make a new survey of the principles of chemistry and to revise all their ideas about the constitution of matter. Lastly, the experiments of the biologists and the

exhumation of Mendel's ideas brought about a crisis of the transformist theories and made it necessary to reexamine all our ideas concerning the evolution of life.

However, if the philosophic revival which is now going on has been principally caused by the progress of science and only began in this century, the movement that slowly prepared it is obviously older and more complex. One must first take into account all the scientific work of the last century. This was perhaps less revolutionary and did not provoke sharp crises, like the discoveries just alluded to, but none the less it obliged scientists to modify and to elevate their point of view. Besides, it must be remembered that the writings of some of the scientists of the nineteenth century, namely Helmholtz, Claude Bernard, Berthelot, were already of a synthetical type. But a philosophic school has also in a great measure contributed to this renaissance: I refer to the positivist school represented in France by Auguste Comte, and in England by Herbert Spencer. Our own endeavors are certainly a direct result of their activity. One might say that the positivist ideas have never been better understood nor more popular than they are now. But we must not be led astray by this. It is only since the progress of science has extenuated at the same time the dogmatism and the agnosticism of the first positivist school, and made its ideals broader and more flexible, that positivism bears all its fruit.

This is the first evolution the explanation of which was necessary to show the origin of our ideas. Resounding discoveries determined very grave crises in many departments of science, and so gave a new scope to the philosophic studies that had been despised for a long time. This new philosophy is simply the old positivism, made more supple and more realistic. This is very remarkable, indeed, because the positivist philosophy that had been built up for the very use of scientists had at first not been able to

triumph over their indifference; its success was not secured until the whole structure of knowledge had been shaken and endangered by the very progress of science.

But this is not all. There is still another crisis that seems to have just reached its climax. The triumph of positivism was a triumph rather for science than for philosophy. Many people thought that philosophy would soon be incorporated into science. It would be a philosophy of science, it would gravitate around scientific facts and ideas, or it would not be at all. Its function would be to think out science, nothing more. Such exaggerations, such a misunderstanding of philosophy's historical role,—namely, to be an independent vanguard, a storehouse of general and leading ideas extracted not only from science but from the whole of human experience,—could not help bringing about a new reaction. This reaction is the intuitionism of Bergson, the radical empiricism of William James, the humanism of F. C. S. Schiller, the instrumentalism of John Dewey. I shall simply call it the pragmatist movement. By loudly asserting the claims of intuition, it asserted at the same time the rights to existence of a philosophy independent of the positive sciences. That is the only point of concern to us. And it is so much the more necessary to lay stress upon it, that, in my opinion, it is the best way to show that the conflict between neo-positivists and pragmatists, if it is partly irreducible, is, notwithstanding that, much less grave than it might appear at first sight. For one thing, we must bear in mind that we have all—philosophers, historians, scientists—the same purpose: we try to explain, to generalize, to deepen, to simplify the data of experience. And our very methods have very close analogies: all our knowledge is to a certain extent scientific knowledge, and the pragmatist himself assumes a scientific attitude when he scrutinizes his intuitions. Moreover, would the deep cause of the conflict between the positivist

and the pragmatist points of view not lie in the very complexity of our intellectual needs? These needs are of a practical, utilitarian nature and at the same time of a theoretical, esthetic nature; we need to think and to understand, but, at the same time, we need to act. Would it not lie also in the complexity of the problems raised by ever changing life? Indeed, does not life sometimes oblige the most determined agnostics to reason like pragmatists, and reciprocally? It is owing to these deep causes, inherent in our own nature and in the nature of things, that these antagonistic points of view evidence themselves and clash during the whole development of human thought. It may be well, indeed, to remember that if the pragmatist theories have appeared in a new and fascinating shape, thanks to the genius of Bergson and James, they are as old as science itself.

It is necessary to make these remarks to show that we have not to trouble ourselves too much about this crisis. Besides, positivists and pragmatists all agree in respecting science and all acknowledge the necessity of knowing it as well as possible and of having continual recourse to it. It is of the utmost concern to all of them to study the principles and the history of science. Therefore, we do not care much for their quarrels; we simply accept and record them as interesting human facts, as a new evidence of our mind's complexity.

In short, scientists and philosophers are at the present time unanimous in wishing that the general tendencies and fundamental principles of science be constantly extricated, criticized and stated with more precision. They are well aware that it is now an essential condition of progress and security. But how will it be possible to conciliate the imperious needs of synthesis and the division of labor?

It would seem that the only possible solution is that which was recommended by Auguste Comte and partly

realized by himself and his disciples: namely, to originate a new great specialty, the study of scientific generalities. To secure the unity of knowledge it will be more and more necessary that some men make a deep study of the principles and of the historical and logical development of all the sciences. Of course, they will not be expected to be perfectly acquainted with all the technical details, but they must have at their command a thorough knowledge of the great lines and of the cardinal facts of each science. It is a very difficult but not an impossible task. The inconveniences of excessive specialization will be happily counterpoised by this new branch of knowledge, which induces a collaboration of philosopher, historian and scientist. It will clearly appear from the following pages that the best instrument of synthesis, and the most natural hyphen between scientist and philosopher is the history of science.

THE HISTORY OF SCIENCE.

Auguste Comte must be considered as the founder of the history of science, or at least as the first who had a clear and precise, if not a complete, apprehension of it. In his *Cours de philosophie positive,* published from 1830 to 1842, he has very clearly brought forward the three fundamental ideas which follow: (1) A synthetic work like his cannot be accomplished without having constant recourse to the history of science; (2) It is necessary to study the evolution of the different sciences to understand the development of the human mind and the history of mankind; (3) It is insufficient to study the history of one or of many particular sciences; one has to study the history of all sciences, taken together. Besides this, as early as 1832, Auguste Comte made an application to the minister Guizot for the creation of a chair, devoted to the general history of sciences (*histoire générale des sciences*). It was sixty years before this wish of his was granted, and the course

entrusted to Pierre Laffitte was inaugurated at the Collège de France in 1892, thirty-five years after Comte's death. Another French philosopher, Antoine Cournot, also contributed to the clearing up of our ideas, namely by the publication in 1861 of his book *Traité de l'enchaînement des idées fondamentales dans les sciences et dans l'histoire.* However the real heir to Comte's thought, from our special point of view, is neither Laffitte nor Cournot, but Paul Tannery. It is hardly necessary to say much of him, because all who have the slightest knowledge of the history of science must needs have come across one of his numerous memoirs, all so remarkable for their originality and exactitude. Paul Tannery himself attached importance to his intellectual connection with Comte and often expressed his admiration for the founder of positivism.

Tannery's philosophy is very different from Comte's, but the greatest difference between them is that Comte's knowledge of the history of science was very superficial, whereas Paul Tannery, being extremely learned and having at his disposal a mass of historical research work which did not exist in the thirties, knew more of the history of science than anybody else in the world. Certainly no man ever was better prepared to write a complete history of science, at least of European science, than Paul Tannery. It was his dream to carry out this great work, but unfortunately he died in 1904.

One can understand the history of science in different ways, but these different conceptions do not contradict each other; they are simply more or less comprehensive. My own conception does not differ much from Tannery's, except that I attach more importance to the psycho-sociological point of view.

Auguste Comte had noticed all the bonds that unite the different sciences, but he did not give enough attention to them. If he had understood that these interactions and this

interdependence have existed in all directions from the very beginnings of science, would not the rigid framework of his *Cours de philosophie* have been burst asunder?

On the other hand, some people seem to think that it is impossible to write the history of science as a whole, that the subject is too great. I should rather say that the very impossiblity is to pick out from this inextricable network the development of one single branch of human knowledge. Moreover it is actually impossible to write the history of any important discovery without writing, willingly or not, a chapter of the history of science—I mean the history of science as a whole. How could we explain, for instance, the discovery of the circulation of the blood if we did not explain the evolution of anatomy, of comparative zoology, of general biology, of natural philosophy, of chemistry, of mechanics? Likewise, to make clear how they succeeded by degrees in determining longitudes at sea, one has to resort to the history of pure and applied mathematics, the history of astronomy and navigation, the history of watch-making, etc. It would be easy enough to give more examples of the same kind.

Further, it is only by considering the history of science as a whole that one can appraise the scientific level of a definite period or of a definite country. It has happened more than once indeed that one science became neglected while others were thriving, or that scientific culture moved from one country to another. But the historian is not deluded by these facts, and he does not think that human genius is suddenly quenched or rekindled; from his synthetical standpoint he sees the torch of light pass from one science to the other or from one people to another. He perceives better than anybody else the continuity of science in space and time, and he is better able to estimate the progress of mankind.

But the historian's mind is not satisfied with the study

of the interactions between the different sciences. He wishes to study also the interactions between the different sciences on one hand and all the other intellectual or economic phenomena on the other hand. As a matter of fact he has to give a great deal of attention to these reciprocal influences, but of course he does not forget that the aim of his work is essentially to establish the interconnection of scientific ideas.

In short, the purpose of the history of science, as I understand it, is to establish the genesis and the development of scientific facts and ideas, taking into account all intellectual exchanges and all influences brought into play by the very progress of civilization. It is indeed a history of human civilization, considered from its highest point of view. The center of interest is the evolution of science, but general history remains always in the background.

It follows from this definition that the only rational way to subdivide this history is not at all to cut it up according to countries or to sciences, but only according to time. For each period of time, we have to consider at once the whole of its scientific and intellectual development.

Of course to make this general synthesis possible, it will often be expedient, or even necessary, to write monographs or partial syntheses of different kinds. For instance, the study of the archives of a definite place leads naturally to the drawing up of an essay on the history of science in that place. On the other hand, a specialized scientist will be tempted to look up the genealogy of an idea in which he is particularly interested, or to write the biography of a forerunner whose work and genius he can better appreciate than anybody else. But all this research is necessarily incomplete and does not acquire its proper significance so long as it cannot be properly inserted into a history of the sciences of the same age. It may be worth while to add that all monographs are not equally useful; some are so

clumsy and absurd that they obscure, mislead and delay the next synthesis.

To elaborate our historical work we have, of course, to use the same methods that are used by ordinary historians to appraise and criticise the materials available to them. But the historian of science has to use, independently, some other methods of a more special nature. I cannot explain them here, but it is easy to understand that, for instance, to establish at what date a discovery became a real part of science and enriched human experience, the historial exegesis must be supplemented by a scientific exegesis, based on the evidence given by the positive sciences.

We must try to marshal all scientific facts and ideas in a definite order; this means that we must try to assign to each of them a date as precise as possible—not the date of their birth or of their publication, but that of their actual incorporation into our knowledge. Likewise biographers have to exert themselves to fix precisely during which periods the influence of great scientists was the most felt, in order to range them in chronological series. That is generally a very difficult thing to do, and the reader will not fail to appreciate the work that is discreetly accomplished by such scholars. This work of erudition is the bed-rock on which all historical writing is built up.

These remarks complete and add precision to our definition of the history of science. However it may be well to give some more details about the different exchanges which the historian has to consider to put the evolution of science in its proper light.

I shall successively examine some of the other departments of life which are the most interesting for the historian of science: (1) General history or the history of civilization; (2) The history of technology; (3) The his-

tory of religions; and (4) The history of fine arts and arts and crafts.

1. *Science and Civilization.* Since the eighteenth century, and notably under the influence of Vico, Montesquieu and Voltaire, the conception of history has become more and more synthetical. History, the principal interest of which consisted in military records and court annals, is growing up into a history of civilization. It stands to reason that a sufficient knowledge of the history of civilization is absolutely necessary, were it only to locate the scientific facts in the very surroundings that gave rise to them.

On the other hand the historian of civilization can no longer remain unacquainted with the history of science. Some of the most recent historical manuals contain paragraphs devoted to it. It is true, the space allowed is rather scanty, but that is a beginning. I feel confident that before long general histories will be written where the history of science, far from being banished to some obscure corner, will be, on the contrary, the very center of the picture. Is not science the most powerful factor of evolution?

Some examples will illustrate the signification of the history of civilization: How can one account for the fact that the Latin manuscripts containing the translations of Greek authors made from Arabic texts, have so long barred the way to the printed translations that had been elaborated direct from the Greek texts? The latter, indeed, were much better. Björnbo has given some reasons that are very probably the true ones. The printed books that nobody cared to copy, became rarer and rarer. On the other hand the manuscripts were copied over and over again and continually multiplied. Besides, the copyists lacked knowledge and critical sense to a great extent, and they could not help being favorably impressed by the bulk of

Arabic literature. Mere scientific reasons do not suffice to explain the creation of the metrical system by the French revolutionaries. This creation was also in part a reaction against the "foot of the king" of the *ancien régime*. Financial or tariff regulations or the promulgation of labor laws can transform the business life of a country and, indirectly, its scientific production. To understand the beginnings and development of geography one has to take into account many facts that are quite foreign to science. For instance: the quest of mythical treasures; conquerors' ambitions; religious proselytism; the adventurous instincts of daring young men. Lastly, it is necessary to know the history of epidemics and to study all the social facts that have been their causes or their results, to correctly estimate the evolution of medical ideas.

2. *Science and Technology.* Industrial requirements are always putting new questions to science, and in this way they guide, so to say, its evolution. On the other hand the progress of science continually gives birth to new industries or brings new life into old ones. It follows that the history of science is constantly interwoven with the history of technology, and that it is impossible to separate one from the other.

Let us see some examples. After exhausting-pumps had been invented there was such a demand for good pumps of this kind that special workshops were founded in the beginning of the eighteenth century, in Leyden, Holland, to make them, and of course these workshops soon undertook to make other scientific instruments. It is hardly necessary to point out how much the making of these instruments is intimately connected with the history of physics or astronomy.

A geological discovery suffices to revolutionize a whole country and transform an agricultural nation into an in-

dustrial one. Of course a transformation as complete as this involves a radical change in scientific needs. The working of mines has always exerted such a deep influence on the evolution of science and civilization that one might compare the importance of mines in the history of science with that of temples in the history of art. L. de Launay has very clearly shown that the silver mines in Laurion played a considerable part in the history of Greece.

The history of chemistry would sometimes be unintelligible if the history of chemical industries was not studied at the same time. Let me simply remind the reader of the case of coloring matters. Industrial research made in this direction has deeply influenced the progress of organic chemistry. On the other hand it is well known how much has been done to improve this industry by the scientists of the German Chemical Society.

A chemical discovery can revolutionize a whole country, just as completely as a geological one; as soon as it becomes possible to realize, on a business basis, the chemical synthesis of a natural product (like indigo, vanilla, India rubber), the agricultural industry and civilization of immense countries will be in danger.

Technical inventions are every day more precisely determined by industrial needs. The manufacturer can often say very definitely to the inventor: "This is the invention which I now need to improve my production." Besides, every invention starts a series of others that the first has made necessary and that it would have been impossible to realize, or even to conceive, before.

Lastly, commercial needs also influence the development of the sciences, not only the natural sciences and geography (that is too obvious to dwell upon), but even mathematics. It is necessary to take into account the evolution of bookkeeping and banking business to thoroughly understand the introduction and the spread of Hindu-

Arabic numerals into Europe, and later the invention of decimal fractions. It is also a great deal owing to commercial requirements that many astronomical discoveries were made, and that the different systems of weights and measures were created.

3. *Science and Religion.* Science and religion never ceased to influence one another, even in our own time and in the countries where science has reached a high degree of perfection and independence. But of course the younger science was, and the farther we go back through the ages, the more numerous these interactions are. Primitive people cannot part scientific or technical ideas from religious ones, or, more exactly, this classification has no sense to them. Later, when the division of labor had created some scientists or engineers, distinct from the priests, or at least had given birth to a class of priests who had undergone a higher scientific training than their colleagues, even then the interpretation of the holy books, the observance of rites, the needs of agriculture and medicine, the making of the calendar, and above all, the hopes, the fears and the anxieties of a very precarious existence, have been innumerable links between science and religion. The great plagues, and generally all cataclysms, for instance earthquakes or wars, have been followed by religious revivals and often by violent outbursts of religious fanaticism.

On the other hand I know many cases where the priests themselves have been the transmitters of knowledge from one generation to the following. The best example of this can be found during the period extending from the end of the second school of Alexandria to the ninth century. We owe, if not the advancement of science, at least its conservation, to the Fathers of the Latin church and to the Nestorian heresy.

In some other cases the influence of religion is less

direct, but not less important. For instance A. de Candolle has proved that the Protestant families which were exiled from the Catholic countries of Europe during the sixteenth and seventeenth centuries and even during the eighteenth, have given birth to an extraordinarily high number of distinguished scientists. That is not to be wondered at. These people who preferred the misery of exile to moral servitude, were certainly above the average as to their conscientiousness and earnestness.

The interactions between science and religion have often had an aggressive character. There has been most of the time a real warfare. But as a matter of fact it is not a warfare between science and religion—there can be no warfare between them—but between science and theology. It is true that the man in the street does not easily differentiate between religious feelings and faith, on one side, and dogmas, rites and religious formalism on the other. It is true also that the theologians, while affecting that religion itself was aimed at when they alone were criticized, have not ceased from aggravating these misunderstandings. An excellent proof of this has been given in this country. One of the great men of these United States, Andrew Dickson White, has published a splendid book on *The Warfare Between Science and Theology.* Mr. White is a very godly man, and his book is, it is hardly necessary to state, extremely liberal and indulgent to everybody. Notwithstanding this, the author and his book had to bear the attacks of a great many fanatics.

One of the saddest results of these misunderstandings is that some very religious and sincere souls have been misled and have treated science as an enemy. Another important result is that the evolution of science is very intimately interwoven with that of religions and their heresies.

4. *Science and Art.* It may be useful to tender some

remarks upon the different characteristics of scientific and artistic work before pointing out what is interesting from our point of view in the history of art.. In the history of art as it is generally taught, very little is said about technicalities. Are there many people who know, or care to know, what kind of colors Botticelli used, or what were the tools of Phidias? We love a work of art for itself. It is essentially the ultimate result that interests us, not the methods employed to obtain it. On the contrary in the domain of learning the result is generally less interesting than the methods employed to reach it.

The history of science is not merely a history of the conquests of the human mind, but it is much more a study of the instruments—material and intellectual instruments —created by our intelligence; it is also a history of human experience. This long experience of the past has much more significance for the scientist than for the artist. The artist admires the work of his forerunners, but the scientist does more than admire, he makes actual use of it. The artist may find an inspiration in it, but the scientist tries to incorporate it entirely in his own work. It is very difficult to conceive progress in art. Does Rodin carve better than Verrochio or Polycletus? The pictures by Carrière, by Watts, or by Segantini, are they finer than those by Fra Angelico, by Van Eyck or by Moro? Have these questions even any sense?

In the domain of science the matter is quite different. Undoubtedly it would be foolish to discuss whether Archimedes was more or less intelligent than Newton, or Gauss; but we can in all security assert that Gauss knew more than Newton, and that Newton knew more than Archimedes. The making of knowledge, unlike that of beauty, is essentially a cumulative process. By the way, this is the reason why the history of science should be the leading thread in the history of civilization. Nothing that has been done or

invented gets lost. Every contribution, great or small, is appreciated and classified. This cumulative process is so obvious that even very young men may be better informed and more learned than their most illustrious forerunners. As a matter of fact they are standing on the shoulders of their predecessors, and so they have a chance to see farther. If they are not very intelligent they may be inclined to think that it is useless to study history, under the misapprehension that they already know of the past all that is really worth knowing. In short, we are not sure that men become more intelligent—that is, whether intelligence increases—but we positively know that human experience and knowledge grow every day. As I have said, one does not pay much heed to mediocre artists. What they do has not much importance. On the contrary, in the laboratories, libraries and museums where science is slowly growing,—like an ever-living tree,—enormous quantities of excellent work is done by thousands of men who are not unusually intelligent, but who have been well trained, have good methods and plenty of patience.

Scientific work is the result of an international collaboration, the organization of which is perfected every day. Thousands of scientists devote their whole lives to this collective work—like bees in a hive—but their hive is the world. This collaboration does not take place simply in space, but also in time; the oldest astronomical observations are still of some use. Perhaps this collective nature of scientific work is one of the causes of the general indifference concerning its history—indifference strongly contrasting with the widespread curiosity about the history of literature and the fine arts. Science aims at objectivity; the scientist exerts himself to decrease to a minimum his "personal equation." Works of art on the contrary are extremely individual and passionate, so it is not to be wondered at that they excite more sympathy and interest.

The history of the fine arts and of literature is generally considered as a history of the great artists and of the works they have bequeathed to us. But one could adopt a different point of view: just as the history of science gives us the materials of an evolution of human intellect, so one could look in the history of the arts and of literature for the story of the evolution of human sensibility. The history of science is a history of ideas; just so the history of art could be considered as a history of man's dreams. Understood in that way, the two histories complete and enlighten one another.

The interactions between science and art have been particularly vivid at the times of naturalistic reactions against scholastic and pedantic excesses. It would be extremely interesting to make a closer study of the rhythm of the different tendencies that swayed plastic arts and music, and to look for similar rhythms in the contemporary succession of scientific theories, or more exactly, attitudes. The interference of some men of genius, who were at one and the same time artists and scientists,—such as Leonardo da Vinci, Albrecht Dürer and Bernard Palissy,—gives us a splendid opportunity to study these interactions in their deepest and most fascinating form. On the other hand it is a fact that scientific ideas have often been transmitted by works of art; moreover for all the period that precedes the beginnings of popular printing these works of art give us direct testimonies—often the only ones we have—of inestimable value. For instance it would be impossible to trace the history of ancient chemistry but for all the works of art and decoration that have come to us; and, to understand the history of chemistry, not only in ancient times but even as far as the threshold of the seventeenth century, it is still necessary to study the development of the arts and crafts,—the art of the potter, of the glassmaker, of the

chaser, of the jeweler, of the miniature painter, of the enameler.

But the history of art helps us, above all, to understand the spirit and the soul of vanished civilizations. From this point of view, works of art have an immense superiority over every other manifestation of the human mind; they give us a complete and synthetical view of times gone by; they offer us the information that we need at a glance; they bring the past to life again. A granite sphinx, a Nike, a picture by Giotto or by Breughel, a Gothic cathedral, a mass by Palestrina—all these things teach us more in one flash that living men could do by long discourses.

The following examples will show what kind of information the history of art can give us. It is by comparing Gothic monuments that Viollet le Duc has been able to find out some of the principal commercial roads of the twelfth century. Illustrations from Roman monuments give us exact information as to the origin of domestic and medical plants. Indeed it is through Greece and Rome that most of them were introduced from the East into Europe. The history of these plants tells us all the vicissitudes that modified the commercial and intellectual relations between those peoples. Here is another very curious fact. The great botanist H. de Vries has discovered the variety *monophylla* of *fragaria vesca* in a picture by Holbein the Elder ("The Saint Sebastian of Munich," dated 1516). This variety is now cultivated in botanic gardens as a rarity. One guesses that similar discoveries, however small they may appear, sometimes accomplish the solution of historical problems.

Lastly, I wish to note that the history of science is also, to a certain extent—perhaps less than some mathematicians think, but much more than the artists suppose—a history of taste. Leaving aside the external beauty of many books of science, for many scientists were splendid

writers (think of Galilei, Descartes, Pascal, Goethe, Darwin), the very substance of their work has often a great esthetical value. Scientists, who are men of taste, very easily distinguish the scientific theories that are beautiful and elegant from the others. It would be wrong to ignore this distinction, because this beauty and harmony, that common people cannot see but that the scientist does see, is extremely deep and significant. One might ask: "These theories that are more beautiful—are they more true?" Anyhow they are easier and more fertile; and for that reason alone it is worth while to give them our preference.

THE SCIENTIFIC POINT OF VIEW.

The history of science has a great heuristic value, especially if it has been worked out by somebody who is well acquainted with modern scientific tendencies as with ancient ones. The sequence of old discoveries suggests similar concatenations to the scientist, and so enables him to make new discoveries. Disused methods, cleverly modified, may be rendered efficient again. When it is understood in this way, the history of science becomes really a research method. A great scientist of our own time, Ostwald, has even gone so far as to say that "It is nothing but a research method." We do not admit this much. Anyhow, new and old science complete and continuously help one another to advance and to diminish the unknown that surrounds us everywhere. Does this idea not illuminate our conception of the universal scientific collaboration? Death itself does not interrupt the scientist's work. Theories once unfolded are eternally living and acting.

To give to our history all its heuristic value, it is not sufficient to retrace the progress of the human mind. It is also necessary to remember the regressions, the sudden halts, the mishaps of all kinds that have interrupted its

course. The history of errors is extremely useful; for one thing, because it helps us to better appreciate the evolution of truth; also because it enables us to avoid the same mistakes in the future; lastly, because the errors of science are of a relative nature. The truths of today will perhaps be considered tomorrow, if not as complete mistakes, at least as very incomplete truths; and who knows whether the errors of yesterday will not be the approximate truths of to-morrow? Similar rehabilitations frequently occur, and the results of historical research often oblige us to admire and honor people who have been misunderstood and despised in their own time. This incidentally proves to us that the study of the history of science has also some moral advantages.

However the history of superstitions and errors must not make us forget that it is the history of truth—the most complete and the highest truths—that interests us primarily. Besides, one may aim at retracing the history of truth in its entirety, because it is naturally limited, but the history of errors is infinite. It is thus necessary to fix some artificial limits to the latter and to choose judiciously between the errors and superstitions. A great simplification is obtained by classifying the errors in groups. Indeed the same mistakes and superstitions appear over and over again in different shapes, and it is useful to know the different types of errors to understand the mechanism of intellect.

It is much to be regretted that many scientists decline to admit the utility of historical research or consider this simply as a kind of pastime of small importance. They base their contempt on the following argument: "All the best of ancient science has been assimilated and incorporated in our own science. The rest did not deserve more than oblivion, and it is awkward to overburden our memory with it. The science that we are learning and teaching

of uncertainty and hypothesis becomes narrower, the oscillations of the mind between divergent points of view are so much less numerous,—but they do not cease entirely. Thus E. Belot has recently reintroduced into cosmology, in a very seductive shape, the vortex theory that one would have thought had been entirely banished by Newton's criticisms. Similarly Walter Ritz has given weighty reasons for reinstating into optics the emission theory, which seemed to have been forever exploded by the discoveries of Huygens, Young and Fresnel.

But the best examples of such return to ancient knowledge are given to us by the history of technology. The history of chemical industries is very significant from this point of view. This is due to the fact that economic conditions here play a considerable part. In order that an invention may be realized it does not suffice that it be theoretically possible; it must pay. Now thousands of circumstances continually modify the material factors which the engineer is struggling with; many are of such a nature that nobody could foresee them, or (what amounts to the same thing), that it would cost too much to insure oneself against all of them. If new products appear on the market, or if the prices of some of the raw materials happen to vary considerably, or if new discoveries are made, or if new residues are to be employed, old methods that were too expensive may become economical, or reciprocally. Hence the chemist and the engineer have a vital interest in knowing the processes that have fallen into disuse, but to which the very progress of science may give from one day to the next a new career. The history of science is to them, so to say, what forsaken mines are to the prospector.

But in my opinion, however important its heuristical value may be, there are still deeper reasons why the scientist should give his attention to the history of science. I am thinking of those which have been so splendidly illus-

trated by Ernst Mach in his *Mechanics*. For one thing, it is obvious that "they that know the entire course of the development of science will, as a matter of course, judge more freely and more correctly of the significance of any present scientific movement than they who, limited in their views to the age in which their own lives have been spent, contemplate merely the momentary trend that the course of intellectual events takes at the present moment."[2] In other words, to understand and to appraise at its just value what one possesses, it is well to know what the people possessed who came before us; this is as true in the domain of science as it is in daily life. It is his historical knowledge that discloses to the scientist his precise attitude toward the problems with which he has to grapple, and that enables him to dominate them.

Moreover while research workers exert themselves to extend the boundaries of science, other scientists are more anxious to ascertain whether the scaffolding is really solid and whether their more and more daring and complex edifices do not risk giving way. Now the task of the latter, which is neither less important nor less lofty than that of discovery, necessarily implies a return to the past. *This critical work is essentially of an historical nature.*[3] While it helps to make the whole fabric of science more coherent and more rigorous, at the same time it brings to light all the accidental and conventional parts of it, and so it opens to the discoverer's mind new horizons. If that work were not done, science would soon degenerate into a system of prejudices; its principles would become metaphysical axioms, dogmas, a new kind of revelation.

That is what some scientists come to, who, for fear of falling into literature or metaphysics (as they put it),

[2] Ernst Mach, *The Science of Mechanics*, translated by Thomas J.McCormack, 2d rev. ed., p. 7. Chicago, Open Court Publishing Co., 1902.

[3] See George Sarton, "Les tendances actuelles de l'histoire des mathématiques," *Isis*, Vol. I, pp. 577-589, especially pp. 587-8.

banish all historical or philosophic considerations. Alas! the exclusive worship of positive facts makes them sink into the worst kind of metaphysics—scientific idolatry.

Fortunately it happens at certain periods of evolution that resounding and paradoxical discoveries make an inventory and a thorough survey of our knowledge more obviously necessary to everybody. We are fortunate enough to be living at one of these critical and most interesting periods.

The purpose of historical criticism is not merely to render science more accurate, but also to bring order and clearness into it, to simplify it. Indeed it is the survey of the past that enables us the best to extricate what is really essential. The importance of a concept appears in a much better light when one has taken the trouble to consider all the difficulties that were surmounted to reach it, all the errors with which it was entangled, in short all the life that has given birth to it. One could say that the riches and fertility of a concept is a function of its heredity, and that alone makes it worth while to study its genealogy.

The history of science is accomplishing an endless purification of scientific facts and ideas. It enables us to deepen them, which is undoubtedly the best way to make them simpler. This simplification is of course the more necessary as science grows bigger and more intricate. By the way, it is thanks to this progressive simplification that an encyclopedic knowledge does not become utterly impossible; in certain cases it becomes even more accessible. For instance is it not easier to study chemistry or astronomy—I mean the essentials of it—now than it was, say, in the fifteenth century?

I think one can infer from all the preceding remarks that no scientist is entitled to claim a profound and complete knowledge of his branch if he is not acquainted with its history. I have compared the scientific achievements

of mankind with the collective work that the bees accomplish in their hives. This comparison is particularly apposite to the scientists who have specialized to excess and diligently work in their own narrow field, ignoring the rest of the world. Such men are doubtless necessary, as are the bees that provide honey. But their endeavors could never give birth to a systematic knowledge, to a science proper. It is the more necessary that other scientists raise themselves above the artificial partitions of the different specialties. Their investigations irresistibly lead them to the study of history, and they obtain from it a deeper apprehension of their own collaboration in the grand undertakings of mankind. Just as one experiences gratification by knowing where one is and why, just the same it gives them pleasure to locate their own task in the world's work and to better grasp its relative import. And also, they understand better than the others do the significance of the thousand and one ties that connect them to their fellowmen —and the power of human solidarity, without which there would be no science.

THE PEDAGOGIC POINT OF VIEW.

In many countries one cannot become a teacher at least in the secondary schools, if one has not studied the history of pedagogics. But is it less important to know the history of what is taught? And will not any one who knows this history be better prepared to distinguish what is essential and really interesting from what is not, and to teach his pupils the best of each science? Moreover will the history of science not enlighten the history of pedagogics?

Science is generally taught in a much too synthetic way.[4] It may be that this method is indeed the best for the average student who passively accepts the master's authority. But those whose philosophical mind is more awake

[4] My experience refers especially to the European continent and to the teaching of the physical and mathematical sciences.

can hardly be satisfied by this food, the preparation of which is unknown to them. Instead of being assuaged by harmonious order and perfect science, they are devoured by doubt and anxiety: "Why does the master teach us so? Why has he chosen these definitions? Why?" Not that they are loath to use synthetic methods; on the contrary, these young men will probably be the first to admire the depth and elegance of such teaching once they have grasped from their own experience its logical appositeness, its generality and its economy. But first of all they want to know "how all that was built up," and their mind instinctively recoils from a dogmatism that is still arbitrary to them.

It remains arbitrary indeed so long as the reasons that justify and render natural one arrangement in preference to any other, have not been explained. I know that it is not easy to teach beginners in this way, but at least the deficiencies of the present methods could be tempered, and I do not ask for more.

Nothing would be more useful from this point of view than to work out some text-books in which science would be expounded in chronological order; this is indeed a very important task for which Ernst Mach has given us some admirable models. These text-books would not be employed for elementary study, unless the pupils used them at the same time as others composed along dogmatic lines. Students should have to study the latter and read the first. But in my opinion, these historical text-books would especially stand professors in good stead, by enabling them to illustrate their lessons and make them more intuitive. Oral teaching, more pliable than written teaching, would easily admit of short historical digressions. Would the students not more easily remember the abstract truths that are impressed upon them in ever increasing quantities, if their memory could lay hold of some live facts?

But that does not exhaust the pedagogic importance of

the history of science. Nothing is better fitted to awaken a pupil's critical sense and to test his vocation than to retrace to him in detail the complete history of a discovery, to show him the trammels of all kinds that constantly arise in the inventor's path, to show him also how one surmounts them or evades them, and lastly how one draws closer and closer to the goal without ever reaching it. Besides, this historical initiation would cure the young students of this unfortunate habit of thinking that science began with them.

Good scientific biographies have also a great educational value; they lead an adolescent's imagination in the best direction. Critical and sincere biographies make excellent contributions to the history of mankind. And would not the students work with a better heart and more enthusiasm, would they not have a deeper respect for science, if they knew a little more about the heroes who have built it up, stone by stone, at the expense of so much suffering, struggle and perseverance? Would they not be more eager to undertake some disinterested research work? Or at least would they not better appreciate the greatness and beauty of the whole if they had, more or less, partaken of the joy and intoxication of seeing it accomplished amidst continuous difficulties?

Lastly, the history of science—even more than ordinary history—is a general education in itself. It familiarizes us with the ideas of evolution and continuous transformation of human things; it makes us understand the relative and precarious nature of all our knowledge; it sharpens our judgment; it shows us that, if the accomplishments of mankind as a whole are really grand, the contribution of each of us is in the main small, and that the greatest ought to be modest. It helps to make scientists who are not mere scientists but also men and citizens.

THE PSYCHOLOGIC AND SOCIOLOGIC POINTS OF VIEW.

The history of science, its birth, its evolution, its diffusion, its progress and regressions, irresistibly imposes upon us a series of psychological problems. We here enter the field of universal history, such as the much lamented Karl Lamprecht has defined it; for the history of science in the main amounts to psycho-sociological investigation.

It is necessary here to make a preliminary distinction. The progress of science is due to two different kinds of causes: (1) Purely psychological causes, the intellectual work of the scientist; (2) Material causes, principally the appearance of new subject matter or the use of improved scientific tools. Of course it is not difficult to show that the origin of these material causes is itself more or less of a psychological nature. But the distinction holds good; a discovery has not indeed the same character, the same psychological importance, if it is the almost automatic result of a technical improvement, as if it is the fruit of a mind's reaction. We propose to discover the general laws of the intellectual evolution of mankind, if such laws exist. These studies might also help us to better understand the intellect's mechanism. But of course we have given up the extravagant idea of establishing *a priori* the conditions of scientific development. On the contrary our end is to deduce them from a thorough analysis of all the accumulated experience of the past.

The best instrument for these studies is the comparative method, and this means that we must not expect to reach a degree of accuracy of which this method does not admit. But no scientific work would be possible in the domain of biology and sociology if one did not have the wisdom and patience to be satisfied with the approximation that is within one's reach. The comparisons may be confined to the realm of science; I would call these the internal com-

parisons. They may also be made between the evolution of scientific phenomena and that of other intellectual or economic phenomena; and these I would call the external comparisons. The greatest difficulty of course is to find evolutionary processes that can be adequately compared and that are sufficiently independent one of another.

The application of this method has already supplied some results which have been very improperly called "historical laws," and the exactitude of which is very variable. The following are some examples which I shall refrain from discussing: Paul Tannery has shown that the development of calculus generally precedes that of geometry. In their choice of decorative elements primitive peoples always pass from animals to plants; they never do the contrary. The hypothesis that has been expressed about the course of civilization from the South and the East to the North and the West, is well known. Remember also the law of historical periods proposed by Lamprecht, and especially the famous law of the three states (*la loi des trois états*), formulated by Auguste Comte. The theory of historical materialism, originated by Karl Marx, which has exerted such a deep influence on the thought of the nineteenth century, is also a proper example.

It is sensible to undertake the study of intellectual activities in the same way as we study the industry of the beavers or the bees. Of the work produced by the human brain we generally know nothing but the results, but these are tangible and can be, if not actually measured, at least compared and appraised with more or less precision. The invention of a machine or the discovery of a natural law, are these not at the bottom phenomena of the same kind as the behavior of a crab or of a sea anemone under determined circumstances? They are, of course, much more complex and their study requires the use of new methods, scarcely explored; but can one not admit, at least as a

working hypothesis, that they do not essentially differ? The psychology of the superior functions of the brain is not necessarily more complicated than that of the inferior functions; I should be rather inclined to think the contrary. For instance would it not be easier to retrace the development of a scientific idea in a clear mind than to disentangle, in the prelogical head of a primitive man, the obscure roots of his instinct of property or imitation?

It is from the comparison of these intellectual facts, as they can be collected by the historian of science from the whole intellectual experience of the world, that we may try to deduce the laws of thought. Human experience has been continuously increasing during the ages, but the intellect itself,—has it evolved? The methods of discovery, the mental experiences, the hidden mechanism of intuition —have they not remained somewhat the same? Is there nothing invariable in men's intellectual behavior? What are those invariants, or at least those relative invariants, those more stable parts of our self? To what extent does the scientific environment exert its influence upon the scientists, and *vice versa*? How do social activities evidence themselves in the domain of science? By what mental processes are the ideas of the initiators integrated in the collective thought, to become, by and by, common notions? All these questions, raised by the history of science, are so many psychological problems.

As to research concerning the psychology of invention, choice materials will be found in the history of technology. The results of technical invention are material objects of a very concrete and tangible nature. Besides, the mechanism of industrial discoveries is especially interesting, because to materialize his ideas the engineer has actually to struggle with all the difficulties of real life. The struggle is more obvious here than in any other domain. It happens that unexpected obstacles are so great that the idea cannot

be carried out; but it also happens very often that the very clash of these obstacles gives birth to new ideas, deeper and richer than the original ones. Then one sees, so to say, the invention gush out from the conflict between matter and spirit.

It would be apposite here to present some remarks about the "genealogical" research work that was initiated by Francis Galton and Alphonse de Gandolle. These very interesting historico-statistical investigations, intimately connected with the eugenic movement, bring new testimonies to the importance of the history of science from the psycho-sociological point of view. But to give a good idea of these studies I should be obliged to make too long a digression from my subject. I simply refer the reader to my previous publications on these matters.[5]

THE HUMANISTIC POINT OF VIEW.

A deeper knowledge and a greater diffusion of the history of science will help to bring about a new "humanism." (I beg the reader to excuse me for using a word that has already been employed in at least two different senses, but I do not find any other that is more adequate to the idea I wish to convey.) The history of science, if it is understood in a really philosophic way, will broaden our horizon and sympathy; it will raise our intellectual and moral standards; it will deepen our comprehension of men and nature. The humanistic movement of the Renaissance was essentially a synthetic movement. The humanists were longing for a new atmosphere and a broader conception of life; their curiosity was insatiable. We have at least this much in common with them. We know also that if science were abandoned to narrow-minded specialists it would soon degenerate into a new kind of scholasticism, without life

[5] George Sarton. "L'histoire de la science," *Isis*, Vol. I, pp. 39-41; also, same author, "Comment augmenter le rendement intellectuel de l'humanité?" *Isis*, Vol. I, pp. 219-242, and pp. 416-473 (unfinished).

or beauty—false and wrong like death itself. This would be another good reason for comparing our task with that accomplished by the former humanists. However their movement was essentially toward the past; ours is much more a movement toward the future.

Science, divided into water-tight compartments, makes us feel uneasy;—a world split into selfish and quarrelsome nations (similar to the Italian and Flemish municipalities of the Renaissance) is too narrow for us. We need the full experience of other countries, of other races; we need also the full experience of other ages. We need more air!

It may be useful to lay some stress on the significance of science from the international point of view. Science is the most precious patrimony of mankind. It is immortal. It is inalienable. It cannot but increase. Does not this priceless patrimony deserve to be known thoroughly, not only in its present state but in its whole evolution? Now most men—most scientists—are unfamiliar with the glorious history of our conquests over nature. Would it not be a great work of peace and progress to bring them to better understand and appreciate this intellectual domain which is privileged among all others, *because it is the only one that is entirely common to all*? Science is not only the strongest tie, but it is the only one that is really strong and undisputed.

Science makes for peace more than anything else in the world; it is the cement that holds together the highest and the most comprehensive minds of all countries, of all races, of all creeds. Every nation derives benefit from the discoveries that have been made by the others. There can be no warfare between high-minded scientists.

The further science progresses, the more its international character asserts itself—and this in spite of all jingo-

ist and imperialist tendencies that may occasionally blind and lower some of its servants.

Just as scientific methods are the basis of well-nigh all our knowledge, just so science appears more and more as the bedrock on which every organization has to be built up to be strong and fertile. It is the most powerful factor of human progress. As Mach has perfectly put it: "Science has undertaken to replace wavering and unconscious adaptation by a methodical adaptation, quicker and decidedly conscious." It is the historian's duty to evidence all the scientific facts and ideas that make for peace and civilization; in this way he will better secure science's cultural function.

The international significance of the history of science has not been thus far better grasped for the simple reason that very few historical studies have been inspired by a real international spirit. For one thing universal histories have been almost exclusively devoted to the achievements of the Indo-Aryan race. Everything in them gravitates round the development of Europe. Of course this point of view is absolutely false. The history of mankind is too obviously incomplete if it does not include, on the same level as the Western experience, the immense experience of the East. We badly need the knowledge and wisdom of Asia. They have found other solutions to our own problems (the fundamental problems cannot but be the same), and it is of the greatest importance to consider these solutions, and to consider them in a humble way. It is a fact that they have very often been nearer to truth and beauty than we. Besides, although the development of the Far Eastern countries has been to a great extent independent of our own, there have been far more exchanges, especially in ancient times, than is generally believed, and it is also of paramount importance to investigate these matters.

The progress of mankind is not simply an economic

development; it is much more an intellectual unfolding, as Henry Thomas Buckle has shown with so much force. The whole course of civilization is marked by the triumph of the mental laws over the physical—a triumph of man over nature. To the best of my judgment Buckle has even gone too far in this direction. I am not ready to concede, as he has done, that the changes in every civilized people are dependent solely on three things: (1) The amount of knowledge of the ablest men; (2) The direction of this knowledge; (3) Its diffusion. If Buckle were right all history would be included in the history of science. There are other things to consider.

Moral factors do not deserve to be despised as much as Buckle did, and I think that it is even possible to construct an ethical interpretation of history. To give a moral significance to history the essential condition is to make it as complete, as sincere as possible. Nothing is more demoralizing than histories *ad usum Delphini*. We must display the whole of human experience, the best and worst together. The greatest achievement of mankind is indeed its struggle against evil and ignorance. Nothing is nobler than this endless struggle between the truth of to-day and that of yesterday. It stands to reason that if one side of the picture is not shown—the evil side, for instance—the other loses a great deal of its interest. The quest of truth and beauty is indeed man's loftiness. This is certainly the highest moral interpretation of which history allows.

We must try to humanize science, to better show its various relations with other human activities—its relation to our own nature. It will not lower science; on the contrary, science remains the center of human evolution and its highest goal; to humanize it, is not to make it less important, but more significant, more impressive, more amiable.

The new humanism—as I venture to call the intellec-

tual movement that has been defined in the preceding pages—will also have the following consequences: It will disentangle us from many local and national prejudices, also from many of the common prejudices of our own time. Each age has of course its own prejudices. Just as the only way to get rid of local prejudices is to travel,—similarly, to extricate ourselves from time-narrowness we must wander through the ages. Our age is not necessarily the best or the wisest, and anyhow it is not the last. We have to prepare the next one, and I hope a better one.

If we study history it is not through mere curiosity, to know how things happened in the olden times (if we had no other purpose than this our knowledge would indeed be of a very poor quality); nor is it for the mere intellectual joy of better understanding life. We are not disinterested enough for that. No; we wish to understand, to better foresee; we wish to be able to act with more precision and wisdom. History itself is of no concern to us. The past does not interest us but for the future.

To build up this future, to make it beautiful, as were those glorious times of synthetic knowledge, for instance that of Phidias or of Leonardo da Vinci, it is necessary to prepare a new synthesis. We propose to realize it by bringing about a new and more intimate collaboration between scientist, philosopher and historian. If that were accomplished so much beauty would be given birth to that the collaboration of the artist would also necessarily be secured; an age of synthesis is always an age of art. This synthesis is what I have called "the new humanism." It is something in the making,—not a dream. We see it growing, but no one can tell how big it will grow.

The writer is convinced that the history of science—that is to say, the history of human thought and civilization in its broadest form—is the indispensable basis of any philosophy.—History is but a method—not an aim.

APPENDIX.

THE TEACHING OF THE HISTORY OF SCIENCE IN THE UNITED STATES.

An elaborate essay on this subject has been published in *Science*, November 26, 1915, pages 746-760, by Frederick E. Brasch ("The Teaching of the History of Science; Its Present Status in Our Universities, Colleges and Technical Schools"). As I shall confine myself to remarks of my own and to only a few extracts from Mr. Brasch's work, the reader who desires to follow up the subject is recommended to read his paper.

To Harvard University belongs the credit of first establishing a course on the history of a particular science: Dr. Theodore W. Richards began as early as 1890, and is still continuing, a course on the history of chemistry. On the other hand the Massachusetts Institute of Technology was the first to recognize the interest of the history of science as a whole: Prof. W. T. Sedgwick and H. W. Tyler have been teaching it in that institution since 1905.

According to Mr. Brasch's painstaking statistics, 162 courses on the history of some special science are now organized in 113 schools. Among them not less than 47 are devoted to the history of mathematics, and not less than 38 to the history of chemistry. Moreover there are 9 courses on the general history of science. To this number could be added 8 temporary courses, namely, Harvard Exchange Lectures, delivered by Dr. L. J. Henderson in five Middle Western colleges, and three courses given by myself at the summer school of the University of Illinois, at the George Washington University in Washington, D. C., and at Clark University.

Mr. Brasch gives the following information about the nine regular courses: (1) Reed College: history forms a part of a course on general science; (2) Lehigh University: "combination of biographies and progress of science"; (3) University of Pennsylvania: the philosophy department has started a historical course entitled "Philosophy of Nature"; (4 and 5) Chicago and Columbia: history of the physical sciences; at the University of Chicago there is a course on the history of science in America; (6 to 9) Harvard, Princeton, the Carnegie and the Massachusetts Institutes of Technology have organized complete courses on the history of the physical and biological sciences.

This information is very meagre. For lectures on a subject that is still so far from being standardized it would be most inter-

esting to know exactly what are in each case the purpose and the methods of the lecturer. It would be interesting also to know how many of these courses have been given by specially trained men and how many have been more or less extemporized by professors already engaged in other fields.

It is worth while to note that Prof. W. T. Sedgwick and H. W. Tyler are preparing a text-book for the use of their own classes. Dr. Walter Libby of the Carnegie Institute of Technology is also preparing the edition of a series of short volumes on the same subject. As the interest in it is now awakening it is likely that many other text-books will appear before long.

I have come to the conclusion that the history of science as a whole, brought at least as far as the eighteenth century and including perhaps some rudiments of this history in our own times, should be taught to all undergraduate students. It would be for them the best scientific introduction, and at the same time it would provide them with a historic and philosophic foundation on which they could build up their special studies. It would open their minds and broaden their horizon from the beginning. Such a course should be taught by some one devoting himself entirely to historical research of this kind. On the other hand the complete history of each science during the last fifty or a hundred years should be studied by all the graduate students, making a special study of the same. This course should be taught by specialists of a quite different kind,—not historians, but scientists, having a sufficient historical knowledge,—generally professors of the school for graduate studies.

It may be objected to my plan that the scientific preparation of most undergraduate students is so scanty that they would not be able to attend these lectures with real profit. In this case it would perhaps be better to reserve them for the graduate students, or to shift them to the very end of the university curriculum. In this second hypothesis the course could be made much more complete and be treated from a much higher point of view. It could be a really inspiring course, giving much food for thought to the best students,—a splendid coronation of their studies. It would open their eyes to the marvelous spectacle of human evolution. It would be for them, before their departure from the university, the great humanistic initiation, the supreme lesson of wisdom, of tolerance and enthusiasm.

Some may doubt whether courses on the history of science are really as necessary as I claim. But one thing is certain: If they are given at all they must be given well. A loose and superficial teaching is worse than none. It would soon bring discredit upon historical studies. We must avoid that at all cost. Therefore it is urgent to organize a seminary in at least one of the universities of this country where normal lessons would be given and the historical methods taught in the experimental way. Those who teach the history of science must needs have a first-hand knowledge of it and be trained to make accurate investigations.

There is no seminary for the history of science in this country, but there is one for the history of mathematics at Teachers College of Columbia University, under the direction of Dr. David Eugene Smith. A splendid library and interesting collections have been formed by him at Teachers College, and original research work on the history of mathematics can be conducted there under the best conditions.

Some seminaries also exist in Europe. I know at least two that are equipped for the study of the history of medicine: the famous Institut für Geschichte der Medizin of Leipsic, so efficiently directed by Dr. Karl Sudhoff, and another one in Vienna under the direction of Dr. Max Neuburger. On the other hand the much lamented A. von Braunmühl founded in Munich a seminary devoted to the history of mathematics; and of course much seminary work was also done in Heidelberg, under Moritz Cantor's direction.

There may be other seminaries which I do not recall; but I know positively that there is none devoted to the history of science as a whole. That is not to be wondered at, as these studies are scarcely begun.

I hope that one of the great American universities will take upon itself this initiative, and organize an institute where all information on the history of science could be centralized, studied and diffused again.

Will America give this great example to the world? I earnestly hope so.

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The John Crerar Library of Chicago published in January, 1911, "A list of books on the History of Science, prepared by Aksel G. S. Josephson." It is the only list of this kind that I know of, and it is very valuable indeed. However it is far from being complete. For one thing it is simply a list of *books*,

and most historical memoirs are not published in book form. I hear that a supplement is being prepared, and also a companion volume on the History of Industry and Industrial Art. I sincerely hope that the Supplement will contain some critical notes, which allow the reader to make a sensible choice between so many titles. Uncritical bibliographies, where the best and the worst books are all put on the same level, sometimes do more harm than good.

The best way to complete the information given by Aksel G. S. Josephson is to refer to the "Bibliographie critique de toutes les publications relatives à l'histoire, à la philosophie, et à l'organisation de la science," published in *Isis*. Unfortunately this publication has been interrupted by the war, and the last list published (Vol. II, pp. 249-310) was closed in May, 1914. Two other lists were prepared, and one was in the press, when Belgium was invaded. The offices of *Isis* are of course inaccessible. But more copies of the periodical are still obtainable from the publisher for Switzerland and Germany: Max Drechsel, Akademische Buchhandlung, Bern, Switzerland.

It may be useful also to refer to the following article: George Sarton, "Soixante-deux revues et collections consacrées à l'histoire des sciences (Bibliographie synthétique...., I), *Isis*, Vol. II, pp. 132-161 (1915).

GEORGE SARTON.

WASHINGTON, D. C.

THE ANTHROPOLOGY OF THE JEW.

WITH respect to no other people has there been so much hair-splitting controversy as regards classification as with the Jews. Antisemites and philosemites, anthropologists and historians, political reformers and sociologists, Jews and non-Jews, friends and foes alike have all differently defined and described this peculiarly persisting element. Some would call them a race, others a people, still others a religious sect, and so forth. Thus with Chamberlain, Dühring, Wagner, Woodruff and other antisemites the Jews are a race, but distinctly inferior to the so-called Aryan race; with Wirth, Topinard, Weissenberg, Fishberg, Neubauer, etc., they exist only as a social-theological organism; others, as Ripley for instance, would not call them a race but a people, who have only one element in common, and that is a peculiar facial expression.[1] Lazare, on the other hand, would not call them a race, which to him is a misnomer, since no races in the sense of ethnic unities exist, but to him they are a nation, in the sense of unity of sentiments, ideas, and ethics.[2] Again, Zollschan, Ruppin, Jacobs, Haupt, Andree, Sombart, Salaman, Lucien, Wolf and others believe in the comparative purity of the Jewish race, at least since the time of Ezra, 430 B. C. Zangwill, in a mood of despair, asserts that the Jews exist only as a negative unity, by force of hostile conditions. He says: "No Jewish people or nation now exists,

[1] W. Z. Ripley, *The Races of Europe*, 1899, pp. 368-400.
[2] B. Lazare, *Antisemitism,—its History and Causes*, 1903, p. 248.

but a multitude of individuals; their only unity being negative; the hostile hereditary vision of the ubiquitous Haman."[3]

In juxtaposition to this is the difficulty of identifying the Jews with any of the great subdivisions of mankind. The old Semitic affiliation has lately been called into question. Von Luschan, Ripley, Lombroso, etc., are inclined to believe that the Jews are more Aryan than Semitic. Von Luschan emphatically asserts that they are composed of three elements,—the Hittite, the Xanthecrous Nordic, whom the present Kurds resemble and who he thinks were affiliated with the Amorites of the Bible, and last the Semitic element; the first two he shows were Aryans.[4] Haupt, like Von Luschan, believes they have descended from the Amorites, Hittites, and Armenians, but that the Hittites may have been of Mongolian origin. He also informs the writer in a personal letter that he believes that not only the Amorites but the Phenicians also came from Europe.[5] Judt, cited by Zollschan, on the other hand, thinks the Jews are to be classed with the Alpine races.[6] Again, there is also the question of the superiority or inferiority of the Jew, which has been so much a point of combat between antisemites and philosemites. Indeed to go into the details of the anthropology of the Jew alone would take us far beyond the scope of this article and would in fact lead us nowhere. We shall content ourselves therefore with establishing a few general facts, and in the light of those facts shall pass the verdict whether or not the Jews are to be considered as a race.

The main fault with the majority of theories lies in their one-sided attitude of partiality. The Jew is not con-

[3] I. Zangwill, *The Jewish Race*, 1911, pp. 268-279; G. Spiller (ed.), *Inter-Racial Problems*.

[4] F. von Luschan, *The Early Inhabitants of Western Asia*, pp. 221-244; *Journ. of Anthrop. Inst. Gr. Br. and Irel.*, N. S., XIV.

[5] P. Haupt, "Die Juden," *Meyers Konversations-Lexikon*, pp. 328-330.

[6] I. Zollschan, *Das Rassenproblem*, 1910, pp. 57-58.

sidered collectively as an integral part of an exceedingly complicated organism which we call mankind, but he is measured generally through the horoscope of one's special line of interest. The scientific antisemite, eager to prove his own superiority, considers only that side of the Jew which is below his own standard, underestimating or completely ignoring other phases in which the Jew is markedly above his standard. The same is true of the philosemite *mutatis mutandis*. So the physical anthropologist considers only the physical side, the economist the economic, the politician the political side, and so forth. Indeed it is only natural to undervalue everything outside of our own line of interest, but none the less faulty. We forget that what makes an individual and a race or people as an aggregate of individuals is an *ensemble* of many things, a totality of physical, psychical, physiological and pathological factors, and it is all of these that have to be considered.

Let us turn now to the above-indicated questions. To begin with the question of the superiority or inferiority of the Jews, we think that the common misconception is partly due to the confusion of the term "inequality" as synonymous with either superiority or inferiority. It is really of inequality of the Jew and non-Jew that we should speak, but inequality does not necessarily mean either superiority or inferiority. We cannot speak of the value of abstract qualities as equal or unequal in the sense of coincidence, as in the case of physical measurements. It is the comparison of the values of those qualities, even though different in kind and nature, that we ought to consider. Two individuals may each excel in one thing; they will be unequal in that their lines of excellence are different, but they are not necessarily superior or inferior to each other, for to society the value of the contributions of each may be of equal importance. In a like manner two races may differ in aptitudes for certain lines of endeavor, but their value

to society may be equal. It is only when a comparison of the value of the sum-total of contributions to civilization of one race has been found in a great measure less than that of the other, as would be in the case of the Australian, for instance, and any of the European races, that we may use the terms inferior or superior. Keeping this in mind, we believe that on the whole in the case of the Jew, intellectually he is neither superior nor inferior to any of the European peoples. The Jews excel in some lines and fall short in others, and so with the other races. On the whole they compare pretty well. This is borne out by Jacobs[1] in his study of the "Distribution of Jewish Ability," showing the comparison per mileage of celebrities of Jews with Europeans. We reproduce it in full:

	EUROPEANS	JEWS
Actors	21	34
Agriculture	2	00
Antiquaries	23	26
Architects	6	6
Artists	40	34
Authors	316	223
Divines	130	105
Engineers	13	9
Engravers	3	9
Lawyers	44	40
Medicals	31	49
Merchants	12	43
Military	56	6
Miscellaneous	4	3
Metaphysics	2	18
Musicians	11	71
Natural Science	22	25
Naval	12	25

[1] J. Jacobs, "The Distribution of Jewish Ability," *Jour. Anthrop. Inst.*, Vol. XV, pp. 351-379.

	EUROPEANS	JEWS
Philologists	13	123
Poets	20	36
Political Economy	20	26
Science	51	52
Sculptors	10	12
Sovereigns	21	12
Statesmen	125	83
Travelers	25	12

This table shows a preponderance of Jewish excellence as actors, doctors, financiers, philosophers, musicians, philologists, poets, a slight excess as antiquarians, in natural science and political economy. They are below in agriculture, novel writing, divinity, engraving, military and naval science, as sovereigns, statesmen and travelers; slightly below as painters, engineers and lawyers. They are about equal as architects, scientists and sculptors. Of course some allowance must be made for the fact that the great bulk of Russian Jewry is practically barred from obtaining eminence on account of political and social oppression, as are also German Jews from entering naval and military professions as well as from statesmanship. It is also seen from this table that Jewish ability tends more in the line of abstract thought, which is partly doubtless due, as pointed out by Jacobs, "to their long life in cities and exclusion from nature on the one side, and from education which lies in handicrafts, on the other."[8]

If we class military and naval under one head, as also sovereigns and statesmen, since they are interdependent, we see that the Jews greatly excel in 7 subjects and are below in 7; they slightly excel in 3 and are slightly below in the same number; they are equal in the others, so that both sides compare equally well. Of course there is another question as to whether the same value is to be attached

[8] *Loc. cit.*

to the different subjects. Should we, for example, rate equally military science and philology, or agriculture and music, or philosophy and statesmanship? But I think we can easily dismiss this difficulty if we only bear in mind that it is all these combined that make up civilization and all are necessary and important links. Considering this, we can, I think, without reserve accord the Jew a place in higher civilization equal to that of any of the so-called Aryan stock.

We come to the second point: Is the Jew a Semite or an Aryan? We can easily dismiss this by simply remarking that the original composition of the Jew is absolutely of no consequence whatsoever. What matters it whether the Jews four or five thousand years ago were Hittites, Amorites, Semites, or a conglomeration of them all? It is not what entered into their their make-up, but what they are *now* that is of importance, and what they are now they are by virtue of a long history and specific phylogeny, the only things that make and create races.

And now as to the first question. Have the Jews a right to be considered as a unity, call it race, people, nation or what not? Or are they simply a heterogeneous mass with no coherence or common elements, as Fishberg's[9] arguments would imply? We must note in the first place that the effect of environment on variation of type will be greatest with the Jews, on account of their scattered condition and frequent wanderings, change of habitat, abnormal social and economic conditions, and so forth. Concerning the effect of environment, an authority like Beddoe is inclined to believe that both pigmentation and the form of the skull are directly influenced by the kind and quality of food, apart from its sufficiency or insufficiency in quantity. Robert Gordon Latham thought that form and color might in some degree depend on the geological structure

[9] M. Fishberg, *The Jews*, 1911.

of the habitat, the abundance of carboniferous limestone favoring development of form and complexion. Durand de Gros finds physical differences between the people of the calcareous and granitic parts of Rouergue (south of France), which he thought cannot be accounted for by difference of race. The inhabitants of the calcareous parts are of better form and complexion, while those living in the granitic country are smaller, inferior in form and complexion, less strong but more active. Excess of phosphate of lime in food seems to conduce to good physical development. Thus in Nidwalden and Ticino, two cantons in Switzerland, are found the most robust men, owing, in Beddoe's opinion, to only one point which they have in common, the consumption of great quantities of cheese, an aliment exceedingly rich in phosphate of lime.[10] Again, Professor Lyde points out that pigmentation is not alone influenced by temperature but also by the amount of humidity in the air, the latter favoring fairness.[11] Sergi believes that the presence of blondness in North Africa, which has been advanced as an argument against the effect of environment, is to be attributed to the influence of altitude. Its center of formation was in the Atlas valleys, especially Morocco, which is a region of perpetual snow and cold, not unlike some Alpine and Apennine valleys. From there he thinks it has spread into the neighboring countries as far as the sea in Algeria and Tunis. Ridolfo Livi finds that in Piedmont, Liguria, Veneto, Emilia, Lombardy, Tuscany, Marches, Lazio, Campania, Basilicata, Calabria, Sicily and Sardinia, beyond 401 meters above sea level the blonds predominate over the brunettes, with the exception of Umbria and Abruzzi. The exception he attributes to the fact that those two provinces are hilly almost throughout, with

[10] J. Beddoe, *The Anthropological History of Europe*, 1912, pp. 34-36.
[11] L. W. Lyde, *Climatic Control of Skin Color*, 1911, pp. 104-108; Spiller, (ed.), *Inter-Racial Problems.*

no marked difference between the small plain regions and the surrounding hills.[12]

Indeed, if these arguments bear any weight at all toward the explanation of fairness and darkness in general, their importance should be greater with regard to the Jews, who have been subjected to all climes and all conditions. The fact that the Jews resemble closely the peoples with whom they live, as is seen from the table given below, confirms rather than disproves the theory of climate. This has been conclusively proven by Boas, who has shown that there is a decided tendency in the offspring of immigrants to approach the native head-form. Surely mixture would not account for this change. The explanation is simple; the aborigines or the first settlers of any country have their head-form shaped by the climate and habitat, and any people migrating into the same country undergo the same change without necessarily mingling in blood. The effect on pigmentation may be the same, but the change is so much slower that it becomes perceptible only after millennia.

Following is a table taken from Fishberg[13] showing comparison of cephalic indices of Jews and their non-Jewish neighbors.

COUNTRY	AVERAGE OF CEPHALIC INDEX OF JEWS	NON-JEWS
Lithuania	81.05	81.88
Roumania	81.82	82.91
Hungary	82.45	81.40
Poland	81.91	82.13
Little Russia	82.45	82.31
Galicia	83.33	84.40

The differences as seen from this table are slight, being greatest in Roumania and Galicia, where it exceeds only

[12] G. Sergi, *The Mediterranean Race*, 1901, pp. 73-75.

[13] M. Fishberg, *The Jews*, 1911, p. 52; F. Boas, "Changes of Bodily Form of Immigr. Desc.," *Abst. of Reports of Immig. Comm.*, Vol. II, 1911, pp. 501-556.

one unit. As intermarriages in these countries are least likely to occur, the probability of the effect of environment in tending to approximate the native head-form is still increased.

But environment is not the only factor that may explain the presence of blonds. Even heredity points that way. Experiments in the inheritance of color tend to show that whereas offspring as a rule do not exceed their parents in intensity of pigmentation, they frequently are of a lighter color, so that darker parents may produce light offspring. Davenport, on investigating the inheritance of hair-color in man, finds that out of 210 children whose parents had black hair 3 had flaxen hair, 4 yellow, 5 yellowish-brown, 8 golden, 60 light brown, 37 brown, 49 dark brown, 40 black, and 4 red. It is seen from this that 156 or fully 74 percent of the total had hair lighter than their parents. Davenport also investigated inheritance of eye-color and hair-form, and combining the results of the three investigations he concludes: "It appears that two parents with clear blue eyes and yellow or flaxen straight hair can have children only of the same type, no matter what the grandparental characteristics were; that dark-eyed and haired, curly-haired parents may have children like themselves, but also of the less developed condition."[14]

Of course it may be argued that these are the results of segregation or alternate inheritance in the F2 generation in the Mendelian sense, but his expectations do not exactly tally with his results and are far from being precise, which he himself admits. It is more likely that the results are due to a slight suppression of the pigment factor, the cause of which may be physiological. In the cases of three plants —the sweet pea, the stock and the orchid—Bateson finds that the production of color depends upon a fortuitous con-

[14] C. G. and G. B. Davenport, "Heredity of Hair Form in Man," *Amer. Nat.*, Vol. XLII, 1908; "Heredity of Eye Color in Man," *Science*, N. S., Vol. XXVI, 1907; "Heredity of Hair Color in Man," *Amer. Nat.*, Vol. XLIII, 1909.

course of complementary factors which are independently distributed in gametogenesis, and individuals lacking either of these factors are entirely devoid of color.[15] In the same way it is possible, if one of these factors is partially suppressed by the influence of some external cause, that colors of a lesser degree of pigmentation will arise.

Lightness of color in offspring, unlike parents, may also be due to variations or mutations in the De Vries sense, not of course resulting as he thinks in the creation of an entirely new type, but in the creation of a new character. Brachycephalism among Jews may be due, as pointed out by Jacobs, to intense mental activty, greater among the Jews than any other people.

No less probable is Salaman's suggestion that the divergence in type may be due to the union of characters in gametogenesis in a way similar to that of Bateson's peas, where two apparently similar white sweet peas when mated together gave rise to a purple pea, and when the latter was interbred it produced a series of purples, reds and whites.[16]

Quite probably, also, blondness among Jews is to be attributed, as Von Luschan and Haupt are inclined to believe, to the original constituents of the Hebrews, the Hittites and Amorites. The objection that Fishberg raises that in that case the proportion of blonds among Jews in all parts of the world would be the same[17] does not seem to us to hold, for it may be due to the unequal distribution of the blond elements, so that one place may have more and another place less than it should have in proportion to the total number of Jews in that place, aside from environmental and other factors that may produce disproportion.

But, on the other hand, if we even admit that mixture is the only cause of diversity of types among Jews, it could

[15] W. Bateson, *Mendel's Principles of Heredity*, 1913, pp. 88-97.

[16] R. N. Salaman, "Heredity and the Jew," *Jour. of Genetics*, Vol. I, 1910-11, pp. 273-290.

[17] M. Fishberg, *The Jews*, 1911, p. 507.

hardly be explained, it seems to me, on the basis of Mendelian segregation, for since less pigmentation is usually recessive to more intense pigmentation, then in the mating of Jew and non-Jew the former will be dominant and the latter recessive as regards color of eyes and hair. Using the Mendelian formula[18] we would have this:

DD × RR gives all DR
DR × RR " IDR : IRR
DR × DD " IDD : IDR
DR × DR " IDD : 2DR : IRR

We must add by way of information for the enlightenment of the general reader that the terms "dominant" and "recessive" as used in Mendelian literature designate the degree of manifestation of one or the other of the individual parental characters in the offspring of two crossed varieties or species, commonly known as hybrid. Hence any character such as size, form, color, etc., which is transmitted entire or almost unchanged in hybridization is termed "dominant," and that which becomes latent in the process "recessive," the latter meaning that the character has either withdrawn or entirely disappeared in the hybrid but may nevertheless reappear again in their progeny. The symbols used to express the relationship of any two pairs of characters are DD "dominant" and RR "recessive" and their combinations, while F1 denotes first hybrid generation, F2 second hybrid generation, and so on.

With this in mind, analyzing the above formulas we see the F1 generation will all appear dominant, in this case of the color of Jewish hair and eyes. When F1 marries again non-Jewish we shall expect the offspring equally divided between light and dark, but we must note that in this case where the non-Jewish marriage occurred for two generations in succession the third generation, which

[18] W. Bateson, *Mendel's Principles of Heredity*, 1913, p. 12.

should contain blond hair and light eyes, will be in the non-Jewish fold. If we take another possible combination, that of DR × DD, in this case hybrid and Jew, the result will be 1DD : 1DR, or all the offspring appearing dark, so that even if the second generation should marry Jewish and become a member of the community the hair and eyes of the third generation will still appear Jewish and the type of the Jews unchanged. If we take the third combination, where two hybrids intermarry, we shall have non-Jewish color of hair and eyes appearing only in the proportion of 1 : 3, but the question is in the first place whether hybrids marrying *inter se* will turn to the Jewish or to the non-Jewish fold, very likely to the latter; in the second place, marriages of hybrids of Jews and non-Jews are least likely to occur, owing, as Salaman pointed out, to the greater choice the hybrid has in finding his mate either in the Jewish community or outside of it. He himself in testing the heredity of the Jewish expression by the Mendelian principle could not find a single example of hybrid mating with hybrid.[19] It is clear, therefore, that the hypothesis of mixture as an explanation for the presence of blond hair and blue eyes among the Jews entirely fails when considered in the Mendelian sense. Surely the number of such cases would be if not *nil*, at least so small that it could produce no perceptible change.

But let us not forget that the problem of heredity of color in man is far from being settled, aside from other considerations, because of the complexity of the transmission of the various color characters. Even Bateson points out that only the inheritance of eye-color alone has been established with any clearness, but with respect to hair-color nothing can yet be said with confidence.[20] The task is much more difficult in the intercrossing of races.

[19] R. N. Salaman, "Heredity and the Jew", *Jour. of Genetics*, Vol. I, pp. 273-290.

[20] W. Bateson, *Mendel's Principles of Heredity*, Cambridge, 1913, p. 205.

Indeed, so many are the factors involved in the inheritance of characteristics in man that no one factor, and the least of all mixtures, can be taken as the only cause. Brinton believes that the variability of traits within the racial limits is an ethnic principle, and that this becomes greater as the race is higher in the scale of organic development. To quote: "No race remains closer to its type than the Austafrican, none departs from it so constantly as the Eurafrican. Wherever we find the unmixed white race we find its blond and brunette varieties, its prognathic and orthognathic jaws, its long-skulled and broad-skulled heads. To establish genealogic schemes exclusively on their differences, as has been the work of so many living anthropologists, is to build houses of cards."

Researches conducted by Virchow, De Candolle, Kollman and others disclosed the fact that in the same city and the same family the children are born brunettes or blonds, dark or light eyed, and to some degree broad or narrow skulled, regardless of their parents' peculiarities.[21] Indeed the writer himself can testify from his own observations, perhaps taking himself as an example, of cases who are of pure Jewish descent, and who can trace their ancestry back for several generations, and who not alone have blond or brown hair but present various ethnic traits in various combinations. But on the other hand the Jews after all are not entirely devoid of common physical characters: they are certainly no more heterogeneous as regards head-form and complexion, the only characters that can be relied on safely in anthropology, than any of the other European races if we except the Jews of Cochin China, the Falashas of Abyssinia and the Samaritans, who in our opinion should not be classed as Jews. Historically the Samaritans have never been part and parcel of the Jewish people; they have not undergone the same shaping and

[21] D. G. Brinton, *Races and Peoples*, 1890, pp. 108-109.

moulding under the same rod by the same forces that have made the Jew as we see him in Europe to-day, even though we admit that they are of common origin, since it is not the genetic but the developmental factors that create a race; not what it was, but what it is. A belief in the Jewish religion alone does not by any means make one a Jew, any more than a negro would be reckoned as belonging to the Anglo-Saxon or Teutonic race because like them he believes in Christianity. It is, besides, a question whether even by origin they could be classed as Jews. Peschel in a footnote emphatically states that the black Jews of Cochin are natives of India, purchased as slaves by true white Jews, and received into the community after the fulfilment of the Mosaic rites.[22] Rohlfs, cited by Jacobs, denies Jewish features even to the Falashas; they are only a negroid element converted to Judaism.[23] The Samaritans are a hybrid people of Jews, Moabites and Amorites, but owing to their complete geographical isolation and practical non-mingling with the other Jews they have not shared in the historical process with the bulk of the other Jews, and cannot properly from a scientific point of view be included in the Jewish race. The same applies to the Karaites, the Daggatauns of the Sahara, the Beni Israel of Bombay, and other tribes in China and elsewhere, which can be reckoned only as religious sects, adhering to the tenets of the Hebrew religion, but not forming part of the Jewish race. The Jews that constitute the Jewish race are those of Europe, Asia Minor and North Africa, and especially those of Russian Poland, Austria and Germany, and the United States, and if we confine ourselves to these, as we should, we shall presently see that they present remarkable uniformity in headform and complexion.

[22] O. Peschel, *The Races of Men*, 1906, p. 11.

[23] J. Jacobs, "On the Racial Characteristics of Modern Jews", *Jour. of Anthrop. Inst.*, Vol. XV, 1886, p. 43.

The following table compiled by Ripley[24] gives the cephalic indices as found by various investigators at different times:

AUTHORITY	PLACE	NUMBER	CEPH. INDEX
Lombroso (1894)	Turin, Italy	112	82.0
Weisbach (1877)	Balkan States	19	82.2
Majer and Kopernicki (1877)	Galicia	316	83.6
Blechmann (1882)	W. Russia	100	83.21
Stieda (1883)	Minsk, Russia	67	82.2
Ikoff (1884)	Russia	120	83.2
Majer and Kopernicki (1885)	Galicia	100	81.7
Jacobs (1890)	England	363	80.0
Jacobs (1890)	England (Sephardim)	51	
Talko-Hryncewicz (1892)	Lithuania	713	
Deniker (1898)	Caucasia	53	85.2
Weissenberg (1895)	S. Russia	100	82.5
Weissenberg (1895)	S. Russia	50 (women)	82.4
Gluck (1896	Bosnia (Spagnoli)	55	80.1
Livi (1896)	Italy	34	81.6
Elkind (1897)	Poland	325 { (men) (women)	81.9 82.9
Deniker (1898)	Daghestan	19	87.0
Ammon (1899)	Baden	207	83.5
Ikoff (1884)	Constantinople	17	74.5

The cephalic indices as seen from this table taken at random among Jews of various countries range from 80 to 83, with the exception of Caucasia, Daghestan and Constantinople, being greatest in Daghestan and smallest in Constantinople, although we cannot attach much weight to these extreme cases, since there the number of observations are so few. From this we excluded Ikoff's observations on 30 Caraims in Crimea with a cephalic index of 83.3, who as we said before cannot be classed properly with the Jews. But what is remarkable is the fact that the observations in Russia, Galicia, Poland, Italy and Baden present the least differences, not exceeding two units which may well be attributed to individual variation. Of all these only .08 are dolichocephalic, while all the rest (fully 99.92 percent) are brachycephalic.

The greatest argument against uniformity of skull is

[24] W. Z. Ripley, *The Races of Europe*, 1899, pp. 368-400.

based on the assumption that the Sephardic Jews, as distinguished from the Ashkenazim, are dolichocephalic. This has never been founded on facts, for the observations made are exceedingly few; but what is more, from such data as is available, even among them the majority are brachycephalic. This is seen from the above table in the case of the Jews from Bosnia and Italy. Jacobs in London finds among the Sephardim about 11 percent even less pure long-headed than among Ashkenazim.[25] Ikoff is the only one who found Sephardim dolichocephalic, but since he observed only 17 crania, no weight can be attached to his results.

Von Luschan made measurements of 1222 Jews, 52 percent of whom were Sephardim of Smyrna, Constantinople, Makri and Rhodes, while the rest were Ashkenazim from Vienna, Austria.[26] Unfortunately he does not give the numbers and indices corresponding to each, but from his curve we find only 47 out of a total of 244, or 19 percent, are dolichocephalic, only 33, or 13 percent, are mesocephalic, while the remaining 68 percent are brachycephalic. Of course we do not know how many of the Sephardim were actually brachycephalic, but the exceedingly small percentage of dolichocephals makes it probable that the majority were brachycephalic. Besides, his curve is faulty in that it contains only one-fifth of the actual number, and we are inclined to think that the author picked out only those that show great variance in head-form, in order to prove the extreme variability of the head-form among Jews, a point which he is trying to bring out. We have no doubt the curve would have been different had the total number been plotted, but even as it is it shows up favorably the other way.

[25] J. Jacobs, "On The Racial Characteristics of Modern Jews", *Jour. of Anthrop. Inst.*, 1886, pp. 23-63.

[26] F. von Luschan, "The Early Inhabitants of Western Asia", *Jour. of the Anthr. Inst. Gr. Brit. and Ire.*, pp. 221-244.

The same uniformity is to be seen from the following additonal figures obtained by other observers:[28]

PLACE	NUMBER	OBSERVER
U. S. Immigr. from Galicia	83.33	Fishberg
U. S. Immigr. from S. Russia ..	82.45	Fishberg
W. Russia	81.05	Fishberg
England	80.00	Jacobs
U. S. Immigr. from Poland	81.91	Fishberg
U. S. Immigr. from Roumania .	81.82	Fishberg
U. S. Immigr. from Hungary ..	82.45	Fishberg
U. S.	81.05	Fishberg
U. S. Immigr. from Persia	81.77	Fishberg
U. S. Immigr.	83.00	Boas
U. S.	81.4	Boas

The difference in all these does not exceed 2, with the exception of England which shows a difference of 3. What is rather remarkable is the exceeding uniformity of all the immigrant Jews in this country, the difference being less than two.

Turning to complexion we find that the brunette type is prevalent, the blonds not exceeding 30 percent anywhere, and being doubtless a result of individual variation. Thus Majer and Kopernicki in Galicia, cited by Ripley, found dark hair to be about twice as frequent as light. Elkind, in Warsaw, finds about three-fifths of the men dark. In Bosnia, Glück found only 2 light-haired men out of 55. In Germany pure brunette types are three times as frequent as light, while in Austria they are twice as frequent among Jewish children as among Christian.[29] Of 60,000 Jewish schoolchildren examined in the latter country only 27 percent had blond hair. In Hungary 24 percent of Jewish children had fair hair, in Bulgaria 22 percent. Of 600

[28] M. Fishberg, *Die Rassenmarkmale der Juden*, 1913, p. 29.
[29] W. Z. Ripley, *The Races of Europe*, 1899, p. 391.

children examined by Fishberg in the schools of the Alliance Israelite in Algiers, Constantine and Tunis only 6 percent had fair hair. Among 4235 Jews observed by the same author[30] in New York the following proportions were found:

	JEWS	JEWESSES
Brunette type	52.62%	56.94%
Blond type	10.42%	10.27%
Mixed types	36.96%	32.79%

This table shows only 10.42 percent pure blonds. In the mixed types are included those who have dark hair with fair eyes or *vice versa,* among whom a large percentage must have been of dark complexion. We must bear in mind that a large majority of children become darker in complexion with growing age. Fishberg also finds in North Africa only 4.62 percent of pure blonds. In Bulgaria Wateff found only 8.71 percent blonds. In Austria again, according to districts, Schimmer found only 8 to 14 percent blonds. "Altogether," in Fishberg's words, "it appears that the proportion of Jews of pure blond type oscillates between 5 and 16 percent, according to the country of birth."[31]

Nor is there any striking difference in complexion between the Ashkenazim and the Sephardim. Jacobs[32] gives the following data:

(*a*)

	LIGHT EYES	NEUTRAL EYES	DARK EYES
290 Sephardim	20	12	68
375 Ashkenazim	27	14	59

(*b*)

	RED HAIR	FAIR HAIR	BROWN HAIR	DARK HAIR	BLACK HAIR
290 Seph. ...	3.5	3.5	15.7	40.0	37.3
375 Ashk. ..	1.1	2.6	17.0	45.6	32.7

[30] M. Fishberg, *The Jews,* 1911, pp. 63-66.

[31] *Ibid.,* pp. 66-68.

[32] J. Jacobs, "On the Racial Characteristics of Modern Jews," *Jour. of Anthrop. Inst.,* 1886, pp. 23-63

The only marked difference between the two, as seen from this table, is in the frequency of erythrism, which is about three times as frequent among the Sephardim, but the percentage however appear to be large, due to the small number observed. If we combine the brown, dark and black, all of which should really be classed as brunettes, and also the red and fair as blonds, we have 93 percent of brunettes among Sephardim against 95.3 percent among Ashkenazim. The eyes show greater difference, but no definite correlation has been established between hair and eyes, and we think that the hair only can be relied upon to designate complexion. Of course the differences vary in different countries, but what is significant is the fact that the dark type is prevalent in both the Ashkenazim and Sephardim.

The prevalence of dark complexion is also borne out by another table taken from Fishberg,[33] which shows the percentage of dark and fair hair among 2272 Jews of New York City.

COLOR	JEWS	JEWESSES
Dark Hair	83.49%	80.17%
Fair Hair	13.98%	16.14%
Red Hair	2.53%	3.69%

Thus we see that even from the strictly anthropological view-point the heterogeneity of type among the Jews is quite small, certainly not enough to ascribe it to mixture, and certainly less than among other peoples. Shall we say that the Teutons, for example, are less heterogeneous, comprising as they do the Saxons and Hanoverians in the north, who speak *plattdeutsch*; the Netherlanders and Flemings of the north of Belgium, who speak Flemish or Dutch; the southern Germans; the Alemanni of German Switzer-

[33] M. Fishberg, "Phys. Anthrop. of the Jews," *Amer. Anthrop.* N. S. 5, 1903, pp. 89-106.

land, Alsace and Baden; the Swabians of Württemberg and Bavaria; the Bavarians of eastern Bavaria and of Austria, who speak *hochdeutsch*; the inhabitants of middle Germany, the Thuringians, Franconians, etc., who speak *mitteldeutsch*; finally, the Prussians, partly composed of Germanized Slavo-Lithuanian elements?

The same is true of the Slavs, among whom, in Deniker's words, "it is useless to look for a 'Slav Type.'" In the east we have the Great, Little and White Russians; in the west the Poles of Russian Poland, western Galicia, Posen and eastern Prussia, the Wends or Sorobes of Saxony and the Prussian province of Saxony, who are undergoing a process of Germanization; the Bohemians of Bohemia and a part of Moravia; the Slovaks of Moravia and Hungary. In the south there are the Slovenes or Slovintsi of Austria-Hungary, the Khorvates of Hungary, the Serbs of Servia, the Morlacks, etc., of Dalmatia; the Herzegovinians, Bosnians, Montenegrins or Tsarnagortsi in other parts of the Balkan peninsula; and finally the Bulgarians, who are of Turco-Finnish origin, but Slavonized for at least ten centuries. And so are all the other European and Asiatic peoples.[34]

But more important than physical characteristics are the physiological, pathological and psychological, which are common to the Jewish people as a whole. Of these we can only mention a few. Thus Jacobs in his studies of Jewish biostatics comes to the following conclusions:

1. "Jews have a less marriage rate, less birth rate, and less death rate than their neighbors, but the less marriage and birth rate are due in large measure to the less mortality of Jewish children. The larger number of children living causes the percentage of marriages and births, really larger as regards adults, to seem smaller when reckoned on the whole population."

[34] J. Deniker, *The Races of Man*, 1900, pp. 339-348.

2. "Jews and Jewesses marry earlier than the surrounding population. Cousins inter-marry more frequently, perhaps three times as often."

3. "Jews have larger families, though fewer plural births. On the other hand, mixed marriages between Jews and persons of other races are comparatively infertile."

4. "In Jewish confinements there are more boys, less still-births, and fewer illegitimate births, though the advantage as to still-births disappears among Jewish illegitimate children."

5. "Jews have a smaller mortality of children under five, but this does not hold of Jewish illegitimate children, who die off at much the same high rate as the unfortunate beings of the same class in other sects. Jewish deaths over sixty are generally greater in proportion. Jews commit suicide less frequently."

6. "It has been frequently asserted that Jews enjoy an immunity from certain diseases, notably phthisis and cholera, but the evidence I have on this point is adverse to the claims. There is some indication that they are liable to diabetes and haemorrhoids, and they have certainly more insane, deaf-mutes, blind, and color-blind persons."[35]

The same results were found by Hoffman, Kolb Bergmann, Legoyt, Dernouilli, Lagneau, Loeb and many others.

Lucien Wolf and Dr. Asher who had several years' experience as surgeons to the Jewish Board of Guardians, affirm Jewish immunity from phthisis. Dr. Asher states that in his experience phthisis among English Jews is unknown. This is substantiated by the statistics in the report for 1859 of Dr. Septimus Gibbon, Medical Officer of Health.[36]

Venereal diseases have been found less frequent among

[35] J. Jacobs, "On the Racial Characteristics of Modern Jews," *Jour. of Anthrop. Inst.*, Vol. XV, 1886, pp. 26-27.

[36] *Ibid.*, pp. 56-61.

Jews. Dr. A. Cohen, late Senior House Surgeon of the Metropolitan Free Hospital, London, gives the following figures:[37]

	MEN		WOMEN		CHILDREN	
	NO.	PERCENT	NO.	PERCENT	NO.	PERCENT
Jews	122	17.8	10	20.0	153	3.3
Others . . .	539	62.0	192	62.6	367	15.8

The percentage of the first two rubrics are those of syphilitic cases, the next two are those of gonorrhea, and the last two are those of congenital syphilis observed in the number of children examined.

Cancer is less frequent among Jews. Jacques Bertillon, cited by Fishberg, gives the following figures showing the deaths from cancer per hundred thousand population for different peoples during the years 1903-1908 in Algiers.

French (native)	40
French (naturalized)	18
Jews (naturalized)	21
Spaniards	33
Italians	38

The figures for Amsterdam given by Dr. J. J. Von Konijnenburg are of the same nature.[38] Thus:

		MEN	WOMEN
Jews	(1898-1902)	60	77
Gen. Population . .	(1897-1902)	90	98

In London during 1898-1900 the proportion per hundred of deaths from cancer to deaths due to all causes among persons over twenty years of age was as follows:

	1900	1899	1898
Jews	6.1	6.5	5.02
General Population	8.4	8.8	6.1

[37] *Ibid.*, pp. 31-32.

[38] M. Fishberg, *Die Rassenmerkmale der Juden*, 1913, pp. 117-118.

Jewish children have everywhere been found to suffer less from diseases of the digestive organs. Thus, in Budapest, Körösi finds the death rate from infantile diarrhea during the period 1860-90 to have been as follows per hundred thousand children under five years of age:

Catholics	4143
Lutherans	3762
Calvinists	3293
Other Protestants	3498
Jews	1442

In Vienna Rosenfeld finds the mortality of Jewish children from diarrhea to be only 61 per one hundred thousand population, as opposed to 137 of Protestants and 186 Catholics. In New York, Fishberg[39] calculated from the reports of the Department of Health that during 1897-99 the annual mortality from diarrhea diseases in the entire city was 125.54 per one hundred thousand population, while in the most congested districts, largely inhabited by Jews, it was only 106.79.

The Jew has been found to be deficient in stature, breadth of chest, and lung capacity, by Jacobs, Majer and Kopernicki, Stieda, Glück and others, but in spite of that his tenacity for life has been unprecedented. Especially is this true in the United States. This may be shown by comparing the vital statistics of the Jews as elaborated by Billings in the census of 1890, with that of the general population.[40] It is also seen from the following table given by Hoffman,[41] showing the death rate per one thousand population in the seventh, tenth and thirteenth wards of New York City, 1890, by place of birth:

[39] M. Fishberg, *The Jews*, 1911, pp. 306-312.
[40] D. G. Brinton, *Races and Peoples*, 1890.
[41] W. Z. Ripley, *The Races of Europe*, 1899, p. 384.

AGES	TOTAL DEATHS PER 1000	U. S. INCLUDING COLORED	IRELAND	GERMANY	RUSSIA AND POLAND (MOSTLY JEWS)
Total	26.25	45.18	36.04	22.14	16.71
Under 15 years	41.28	62.25	40.71	30.38	32.31
15 to 25 years.	7.55	9.43	15.15	7.14	2.53
25 to 65 years.	21.64	25.92	39.51	21.20	7.99
65 and over ...	104.72	105.96	120.92	88.51	84.51

As seen from this table mortality among Jews is a little less than one-third of the general population, a little less than one-half for persons under 15, about one-fourth for ages 15 to 25, less than one-fourth for ages 25 to 65, only slightly less however for ages 65 and over.

Considering Jewish biostatics as a whole it would seem to indicate that the Jews are physiologically and pathologically superior to their neighbors, except as regards insanity, deaf-mutism and blindness, but even in these Billings's figures show up favorably. Thus, for example, the proportions per one hundred thousand of population of insane and idiots reported among Jews is 44.5, while that reported by the United States census of 1880 was 336.6, and by the Massachusetts census of 1885 it was 355.3. For deaf-mutes among Jews the proportion was 31.3 per one thousand as opposed to 67.5 by the tenth census of the United States, and 42.6 by the Massachusetts census of 1885.[42] This is not however borne out by all the European figures, and may be due, as Billings suggests, to incorrect reports.

Of course there are differences as to locality, and some investigators confining themselves entirely to one locality are inclined to deny or add one or the other characteristic,

[42] J. S. Billings, *Vital Statistics of the Jews in the U. S.*, Vol. I, No. 19, pp. 1-23.

but by far the greater number of the reports show these differences to be characteristic. Whatever the causes may be, whether dietary laws, family hygiene, beautiful home life, or a result of a long process of selection, the fact remains that those characters are met with in practically every Jewish community.

Intellectually, we have also seen in the beginning of this essay how Jewish intellect tends in a particular direction. This doubtless may be a result, as Lazare points out, of a long continued study of the Torah and Talmud, which shaped the Jewish brain and gave it a characteristic type.[43]

And finally we come to what we consider the most important factor, namely the psychic personality of the race. We have marshalled up all this evidence thus far, and argued both positively and negatively, in order to show that no matter from what angle you approach the problem, whether environmental, hereditary or physiological, the arguments are in favor of the comparative purity of the Jewish race. By this we do not mean that the Jews have abstained from intermarriage, but rather that, by the nature of the facts, the Jew not being of the dominant race and considered more or less a stranger, the majority of those that intermarried left the Jewish fold forever, and the exceedingly small percentage that remained in the community could not possibly affect the Jewish type, certainly not to any noticeable extent. But let us reiterate, After all, of what import is mixture or non-mixture? Ethnically there is no pure race. The old polygenistic view has long been abandoned by men of science. It is conceded by anthropologists that the modern races have not sprung up independently, but have had a common origin. It is not the origin however but the phylogenetic development that a group of individuals, irrespective of its primary constituents, undergoes that finally moulds it into a distinct

[43] B. Lazare, *Antisemitism,—Its History and Causes*, 1903, p. 256.

unit, or what we commonly call race. It is the complete assimilation and fusion of the constituents as a result of long periods of in-breeding and subjection to similar conditions and customs that makes the race. The *summum bonum* of the phylogeneticism is the psychic personality, the soul or race consciousness, if you choose, of each race; and if this is true of any people it is especially true of the Jews, who have tenaciously displayed it in the face of all opposition, with no political boundaries and no center of their own. The characteristic Jewish expression, which even Ripley, Fishberg and Weissenberg do not deny, is, as Fishberg thinks, "the expression of the Jewish soul";[44] but, unlike him, we maintain that it is the most potent, determining factor for each and every race, that it is by far the best guide for distinguishing one race from the other; and while physical characters fail, being as they are subject to environment, physiological, and other changes, it persists in spite of all outward changes. That this is so with the Jews is remarkably affirmed by Salaman's study, who found it to Mendelize, and whose results we give here:

FIRST GENERATION

NO. OF FAMILIES	FATHER	MOTHER	CHILDREN GENTILES	CHILDREN JEWS	CHILDREN INTERMEDIATES
50	Gentile	Jewess	88	15	4
86	Jew	Gentile	240	11	4
Total 136			328	26	8

Out of a total of 136 families tested, of which 50 consisted of father Gentile and mother Jewess, and 86 father Jewish and mother Gentile, there were 328 offspring of Gentile appearance, 26 Jewish, and 8 intermediates; but since the intermediates were more Gentile-looking than

[44] M. Fishberg, *The Jews*, 1911, p. 165.

Jewish, the result is 336 Gentile-looking against 26 Jewish, or the ratio of 13.1. The Mendelian expectation which should have given absolute dominance is short by one, which Salaman attributes to the bias of his observers, who, being zealous Jews, may have taken non-Jewish-looking for Jewish, but what is more probable is the possibility that the non-Jewish parent had Jewish blood. This he actually found to be the case in one family, whose pedigree we reproduce here. All the other families refused to give their genealogies. It may also be due to incomplete dominance, which is quite prevalent even among lower animals and in plants.

Of mating of hybrid and hybrid, Salaman could not find any cases, but he tested 13 families, of which 9 were matings between hybrid mother and Jewish father, and 4, hybrid father and Jewish mother. They had a total of 15 children Gentile-looking and 17 Jewish, as is seen from the following table:

NO. OF FAMILIES	FATHER	MOTHER	CHILDREN	
			GENTILE	JEW
9	Jew	Hybrid	13	12
4	Hybrid	Jewess	2	5
Total 13			15	17

The results as seen from the table fall short of expectation only in two cases. In a personal communication with Dr. Salaman he informs us that he has now additional data which bear out the same results, but which he has not published on account of the war. The results show clearly that the Jewish facial expression behaves as a recessive character to the Gentile, but that it is hereditary just the same. On the other hand the non-Jewish appearance frequently met with among Jews was found by

Salaman to behave as recessive to the pure Jewish appearance.[45]

The persistence of the Jewish type is also beautifully illustrated in Galton's composite photograph compounded of a number of photographs of Jewish boys from the Jews' Free School, London.[46] The typical Jewish expression is remarkably displayed.

It is quite clear that the facial expression of the Jew is a true character, and that therefore the inner psychic personality of the race, of which it is only the outward manifestation, is likewise true and fundamental. The question has been raised as to what has caused the Jewish expression. Some think it is largely a result of long exile and social isolation, as Jacobs suggests; Ripley thinks it is a matter of artificial selection; Fishberg thinks much of it is due to the Jewish costume, etc. But if we keep in mind that the race is the totality of all the elements that have played a part in its history, we can easily see that the expression is a reflection of all the forces that shaped the destiny of the Jewish people. It is neither the result solely of Ghetto life, least likely is it a result of artificial selection, nor can dress and social surroundings change it; they may make it less accentuated, but the features cannot be demolished. In a word, it is not, in our mind, the result of any one thing, but it is a fusion of all the elements that made the Jew as we know him to-day. If we were asked to give those elements we would name them as follows: The sublimity and righteous indignation of the prophets and scribes; the pathos and tragedy of ages of persecution and martyrdom; the cunning and shrewdness that is characteristic of all people who have to live by their wits; a shade of anger or resentment. Finally, we see in the

[45] R. N. Salaman, "Heredity and the Jews," *Jour. of Genetics*, Vol. I, 1910-11, pp. 273-290.

[46] J. Jacobs, "On the Racial Characteristics of Modern Jews, *Jour. of Anthrop. Inst.*, Vol. XV, 1886, pp. 23-63.

their accustomed conquests which are definitive and which they will never need to abandon."

What then is this logistic which made such extravagant claims in 1901 and in 1909 was dead? In order to understand it we must go back to the third century B. C. when Aristotle was developing the study usually called logic. The logic of Aristotle is well enough defined when it is called the logic of classes. A class may be defined in the following terms: Let us suppose that we start with a proposition about some individual, as for example, "8 is an even number," or as another case, "Washington crossed the Delaware." If now we remove the subject and substitute the empty form x, we shall have the statements: "x is an even number, x crossed the Delaware," which are called propositional functions, from analogy to mathematical functions. In this case the functions have but one variable or empty term, x. If we let x run through any given range of objects, the resulting statements will be some true, some false, some senseless. Those that are true or false constitute a list of propositions. For example we may say: "6 is an even number, 9 is an even number, this green apple is an even number," the first a true proposition, the second a false proposition, the third an absurdity. So I might say: "Washington crossed the Delaware, the Hessians crossed the Delaware, the North Pole crossed the Delaware," which are respectively true, false, and absurd, the first two cases being propositions. The propositional function with one variable is called a concept. The individuals that may be put into the empty term (which may be any word of the statement), the variable, and yield true propositions constitute the class of the concept. Thus the class of even numbers consists of a certain endless set or range of individuals, the class of presidents of the United States a certain set of a few individuals, the President of the United States of one individual, and the class of simple

LOGISTIC AND THE REDUCTION OF MATHEMATICS TO LOGIC.

IN the year 1901 we find in an article by Bertrand Russell:[1] "The nineteenth century which prides itself upon the invention of steam and evolution, might have derived a more legitimate title to fame from the discovery of pure mathematics....One of the chiefest triumphs of modern mathematics consists in having discovered what mathematics really is.... Pure mathematics was discovered by Boole in a work which he called *The Laws of Thought*.... His work was concerned with formal logic, and this is the same thing as mathematics."

Also in Keyser's address[2] we find: "....the two great components of the critical movement, though distinct in origin and following separate paths, are found to converge at last in the thesis: Symbolic Logic is Mathematics, Mathematics is Symbolic Logic, the twain are one."

On the other hand we find Poincaré[3] saying after his various successful attacks on logistic: "Logistic has to be made over, and one is none too sure of what can be saved. It is unnecessary to add that only Cantorism and Logistic are meant, true mathematics, those which serve some useful purpose, may continue to develop according to their own principles without paying any attention to the tempests raging without them, and they will pursue step by step

[1] *International Monthly*, 1901.
[2] *Columbia University Lectures.*
[3] *Science et méthode*, p. 206.

Aristotle not only studied classes, with schemes for definition and subdivision of classes, but he introduced the syllogism as a means of reasoning. The syllogism is a succession of three statements of the inclusions of classes; in formal statement, Greek letters denoting classes,

$$\alpha\cdot(\cdot\beta,\ \beta\cdot(\cdot\gamma,\quad \text{then}\quad \alpha\cdot(\cdot\gamma.$$

For example, Pascal's theorem is true of any conic, every circle is a conic, whence Pascal's theorem is true of every circle. For an individual circle we should have a different type of syllogism, a distinction not noted by Aristotle, namely

$$x\ \varepsilon\ \alpha,\quad \alpha\cdot(\cdot\beta,\quad \text{then}\quad x\ \varepsilon\ \beta.$$

For instance, Pascal's theorem is true of circles, this figure is a circle, thence Pascal's theorem is true for this individual circle.

Logic rested with the Aristotelian development for many centuries, and was supposed to be perfect. The regeneration of the subject has been ascribed to Leibniz, because he hoped to see a universal symbolism which would enable the complete determination of all the consequences of a given set of premises to be easily carried out, just as mathematical formulas enable us to solve large classes of problems. This was his Universal Characteristic. But it was reserved for a later day to bring to light the symbolic logic, and we may pass at once to Boole[5] and the nineteenth century. We shall find however in the invention of Boole and his successors not the discovery of mathematics but the mathematicising of logic. The mind again devises new forms for its own use, new ideas by which to attack its problems.

Boole used letters to express classes, the conjunction

[5] *The Mathematical Analysis of Logic*, 1847; *An Investigation of the Laws of Thought*, 1854.

of two letters indicating the largest common subclass, and the formal addition of two letters the smallest common superclass. Then the laws of logic are stated by the formal equations

$a=aa$, (identity); $a+ab=a$, $a(a+b)=a$, (absorption); $ab=ba$, $a+b=b+a$, (commutation); $aa=a$, $a+a=a$, (tautology); $ab=aba$, $a=a(a+b)$, (simplification); $a=ab$, $a=ac$, then $a=abc$, (composition).

He introduced two constants called logical constants, represented by 1 and 0, with the meaning for 1, the minimum superclass of all classes considered, the logical universe; and for 0, the greatest common subclass of all classes, the null-class, or class of impossibilities. It is understood that if a class is considered, the negative of the class is also under consideration, represented by a'. If only one class is considered then $1=a+a'$. If two are considered $1=ab+ab'+a'b+a'b'$, etc. It is evident that

$$1a=a,\ 1+a=1,\ 0a=0,\ 0+a=a.$$

The invention of these notions which seem simple enough now was a great advance over the logic of Aristotle. It suggested for example the use of $1-a$ for a', with the formulas corresponding to algebra

$$a(1-a)=0,\ 1=a+(1-a),$$

the laws of contradiction and excluded middle. Any class may be dichotomized in the form

$$x=ax+a'x=abx+ab'x+a'bx+a'b'x=\ldots$$

If x is a subclass of a we indicate it by the equations

$$x=ax \text{ or } xa'=0.$$

The syllogism takes the very simple form

$$a=ab,\ b=bc, \text{ then } a=abbc=abc=ac.$$

We have thus invented a simple algebra which, with the one principle of substitution of any expression for a letter which the letter formally equals, and the reduction of all expressions by the laws of the algebra, enables us to solve easily all the questions of the older logic. Jevons[6] has stated the rule for doing this very simply: "State all premises as null-classes, construct all necessary subclasses by dichotomy, erase all combinations annulled by the premises, and translate the remaining expressions, by condensation, into the simplest possible equivalent language."

Boole however made a further most important discovery, that there is a nearly perfect analogy between the calculus of classes and the calculus of propositions. That is, we may interpret the symbols used above as representing propositions, under the following conventions. If a is a proposition, a' is the contradictory proposition, ab a proposition equivalent to the joint assertion of a and b, $a + b$ the assertion of either a or b or both, 1 a proposition asserting one at least of all the propositions and their contradictories under consideration, and 0 a proposition asserting all the propositions and their contradictories simultaneously, that is, 1 asserts consistency, 0 inconsistency. A series of formal laws may now be written out and interpreted similar to those for classes. The syllogism, for instance, is the same,

$a = ab$, $b = bc$, then $a = ac$; or in equivalent forms,
$ab' = 0$, $bc' = 0$, then $ac' = 0$.

That is, if the assertion of a is equivalent to also asserting b, and if the assertion of b is equivalent to also asserting c, then the assertion of a is equivalent to the assertion of c. We may reduce the whole scheme of deduction as before to a system of terms which are the expansions of the possible list of simultaneous assertions, the premises annulling

[6] *Principles of Science*, also *Pure Logic*. See also *Studies in Deductive Logic*. Also Couturat, *Algèbre de la logique* (*Algebra of Logic*, translated by Robinson).

certain of these, and those remaining furnishing the conclusions. We should however note carefully that what we arrive at in this manner are not truths or falsehoods but consistencies and inconsistencies. That is to say, we do not prove anything to be true or false by the logic of propositions, we merely exhibit the assertions or classes with which it is consistent or compatible, or the reverse. In this sense only does logic furnish proof. It is obvious however that many new combinations of the symbols used are possible by these methods, and thus it is easy to ascertain the consistency of assertions that would not otherwise occur to us. While the premises evidently are the source of the conclusions, the conclusions are not the premises, and on the one hand the transition from the one to the other is made most easily by these methods, and on the other hand the conclusions are new propositions consistent with the premises. A simple example will show what is meant:

If a implies a', then a is 0; for if $aa=0$, at once $a=0$.

Conv. if $a'a'=0$, $a'=0$, $a=1$.

That is, a proposition which implies its contradictory is not consistent.

It should be noted that the calculus of propositions is not wholly parallel to the calculus of classes. This is shown particularly in the application of a certain axiom, as follows:

$(a\varepsilon \text{ true})=a$ Ax. $a'=(a'\varepsilon \text{ true})=(a\varepsilon f)$. This is absurd for the logic of classes, since $a=1$ is a proposition and not reducible to a class.

A useful form for implication is

$$(a \text{ implies } b)=(a'+b=1).$$

The next advance was due to C. S. Peirce,[7] who devised

[7] *Mem. Amer. Acad. Arts and Sciences*, N. S., IX, 1870, pp. 317-378.

the logic of relatives, in which the propositional function with two variables appears, and which may readily be generalized into the propositional function with any number of variables, giving binary, ternary, and then *n*-ary relatives. As simple examples we may omit individuals that satisfy the proposition: A is the center of the circle *c*, arriving at the propositional function: x is the center of y; or another example with four variables is found in: x is the harmonic of y as to u and v. The calculus of the logic of relations is obviously much more complicated than the previously known forms of symbolic logic. While some of the theorems and methods of the calculus of classes and propositions may be carried over to the calculus of relations, there are radical differences. For instance the relation $x\mathrm{R}y$ is the converse of the relation $y\mathrm{R}x$. These two relations are not identical unless R is symmetric. Again from $x\mathrm{R}y$, $y\mathrm{R}z$, we can infer $x\mathrm{R}z$ only if R is transitive. The ranges of a relation are the sets of individuals that satisfy the propositional function, when inserted for some one of the variables. The most complete development of these notions is to be found in Whitehead and Russell's *Principia Mathematica*. In the intoxication of the moment it was these outbursts of the mind that led Russell into the extravagant assertions he made in 1901. In the *Principia* there are no such claims. It should be noted too that the work of Whitehead in his *Universal Algebra* (1898) contained a considerable exposition of symbolic logic.

As soon as the expansion of logic had taken place Peano undertook to reduce the different branches of mathematics to their foundations and subsequent logical order, the results appearing in his *Formulario*, now in its fifth edition. In the *Principia* the aim is more ambitious, namely to deduce the whole of mathematics from the undefined or assumed logical constants set forth in the beginning. We

must now consider in a little detail this ambitious program and its outcome.

The basal ideas of logistic are to be found in the works of Frege, but in such form that they remained buried till discovered by Russell after he had arrived himself at the invention of the ideas independently. The fundamental idea is that of the notion of function extended to propositions. A propositional function is one in which certain of the words have been replaced by variables or blanks into which any individuals may be fitted. This isolation of the functionality of an assertion from the particular terms to which it is applied is a distinctly mathematical procedure, and entirely in line with the idea of function as used in mathematics. It enabled us above to define concept and relation, in a way, and it further makes quite clear in how great a degree mathematical theorems are about propositional functions and not about individuals. For instance, the statement, "If a triangle has a right angle it may be inscribed in a semicircle," merely means

> right-angled-triangularity as a property is inconsistent with non-inscribability-in-a-semicircle as a property.

In this mode of statement it is apparent to every one that a large part of mathematics is concerned with the determination of such consistencies or inconsistencies. That it is not wholly concerned with them however is also quite apparent. For example, the calculation of π can only be called a determination of the figures consistent with certain decimal positions by a violent straining of the English language. And again, the determination of the roots of an equation is a determination of the individuals which will satisfy a given propositional function, and not a determination of the other functions consistent or inconsistent with that first function. There is a difference well known to

any mathematician between the properties of the roots of a quadratic equation and the properties of quadratic functions of x. Again, the analysis of the characteristics of a given ensemble is a determination of the essential constituents of the propositional function whose roots are the individuals of the ensemble. Operators considered as such are not propositional functions, and neither are hypernumbers. It has been made quite clear, we hope, in what precedes, that much of the mathematician's work consists in building up constructions, and determining their characteristics, and not in considering the functions of which such constructions might be roots. There is a difference between the two assertions

> $2+3=5$ and, If 2 is a number, and if 3 is a number, and if 2 and 3 be added, then we shall produce a number which is 5.

We find the difference well marked in the logistic deduction of the numbers one and two. The deduction is as follows:

Let us consider the propositional functions

"$x \varepsilon \varphi_1$ has only roots such that they cannot be distinguished," as likewise $x \varepsilon \varphi_2, \ldots$ For instance let $(\,)=6$, of which the roots are $4+2$, 2×3, $12/2, \ldots$ which are all indistinguishable in this propositional function. So also $(\,)=9$, $(\,)=4/3\ldots$. Then if we call these propositions similar, in that each *has indistinguishable roots,* we may consider next the propositional function $p\,sim\,(\,)=6$, where p is a variable proposition, which however is distinguished by the character of indistinguishable roots. We may now define the number 1 as the functionality in this functional proposition. That is to say, 1 is a property of propositional functions—namely, that of uniqueness of their roots. In mathematical language we might say: The character which is common to all equations of the form $(x-a)^n=0$, is called *one*, thus defining *one*. Now while it is true perhaps,

that to seize upon equations with one root as cases in which oneness appears, is a valid way to arrive at *one*, nevertheless it is not at all different from any other case in which oneness occurs, as in selecting one pencil from a pile of pencils. In a like manner two is defined as the common property of propositional functions which are relations with a twofold valence, that is, admit two series of roots, the series in each case consisting of indistinguishable individuals. The truth of the matter is that the definitions given are merely statements in symbolic form of cases in which the number one or the number two appears. The two numbers have in no wise been deduced, any more than a prestidigitator produces a rabbit from an empty hat, but they have first been caught, then simply exhibited in an iron cage. The fact that functions are useful things we cheerfully admit, but that everything is reducible to logical functions we do not admit. The arithmetic of 2 and 1 was known long before logistic.

Another notion introduced by logistic is that of truth and truth-value. In no place are either of these terms made clear, nor are they defined. They are qualities of *propositions*, that is propositional functions which have had individuals inserted for the variables. For example, if I consider the propositional function x is right-angled, and then for x insert respectively the triangle ABC, the parallelogram S, this pink color, I have the propositions ABC is right-angled, the parallelogram S is right-angled, this pink color is right-angled. The first of these is said to have the truth-value *truth*, the second the truth-value *false*, the third has the value *absurd*, which is not a truth-value. The first two assertions are then propositions, the third is not a proposition. Much is made of the idea of truth-value, but practically it amounts only to saying that an assertion is a proposition only when it can be labeled with one of two given labels. If any other label is neces-

sary it is not a proposition and not within the region of logistic. So far as really used in logistic these labels are neither more nor less than labels of consistency and inconsistency. They do not refer in any way to objective truth. Thus if we start with the postulates of Euclidean geometry we arrive at certain propositions, as, "triangle ABC has the sum of its angles equal to two right angles." This proposition is not to be tagged as true, but merely as consistent with the premises we started with. The determination of the primitive truth of the premises is not possible by logistic at all. The whole of science is of this character, the truth of the conclusions of science being only probable, not certain, although the reasoning is valid. Science draws its validity from the agreement of all its conclusions with experience. In the same way the conclusions of mathematics are consistent under our notions of consistency, but neither true nor false on account of the reasoning. And this is all that Russell is privileged to say when he asserts that "mathematics is the science in which we do not know whether the things we talk about exist nor whether our conclusions are true." From the results of logistic we certainly do not know either of these things. We merely know that if they exist, and if the premises are true, then the conclusions are true provided the processes of logistic can give true conclusions.

Since logistic does not touch the natural world, and since every one admits that mathematics does give us truth, the only possibility left to Russell was to assert the existence of a suprasensible world, the world of universals of Plato, in another form. In mathematics, he says, we are studying this world and making discoveries in it. It exists outside of the existence of any individual mind, and its laws are the laws of logistic naturally. That such world exists we will readily admit, yet we deny that it stands finished as a Greek temple in all

its cold and austere beauty, but that it is rather a living organism similar to the earth in geologic times, and out of the stress of temperature and moisture and dazzling sun there is evolved through the ages a succession of increasingly intricate and complex forms. But these forms derive their existence from the push and surge of the human mind beating against the cliffs of the unknown. Even logistic itself is the outburst of the mind from the barriers of the early attempts to think and to think clearly. Mathematics finally attacked even the process of thinking, just as it had considered number, space, operations, and hypernumber, and created for itself a more active logic. That this should happen was inevitable. Says Brunschvicg:[8]

"Symbolic logic, like poetic art following the spontaneous works of genius, simply celebrates the victory or records the defeat. Consequently it is upon the territory of positive science that the positive philosophy of mathematics should be placed. It gives up the chimerical ideal of founding mathematics upon the prolongation beyond the limits imposed by methodical verification itself of the apparatus of definitions, postulates, and demonstrations; it becomes immanent in science with the intention of discerning what is incorporated therein of intelligence and truth."

The philosophic assumption at the root of the view taken by the supporters of logistic as the sole source of truth we are not much concerned with, since we are not discussing philosophy but mathematics. But we may inspect it a little with profit. This assumption is the very old one, that there is an absolute truth independent of human existence and that by searching we may find it out. Says Jourdain:[9]

"At last, then, we arrive at seeing that the nature of

[8] *Les étapes de la philosophie mathématique*, p. 426.
[9] *Nature of Mathematics*, p. 88.

mathematics is independent of us personally and of the world outside, and we can feel that our own discoveries and views do not affect the truth itself, but only the extent to which we or others can see it. Some of us discover things in science, but we do not really create anything in science any more than Columbus created America. Common sense certainly leads us astray when we try to use it for purposes for which it is not particularly adapted, just as we may cut ourselves and not our beards if we try to shave with a carving knife; but it has the merit of finding no difficulty in agreeing with those philosophers who have succeeded in satisfying themselves of the truth and position of mathematics. Some philosophers have reached the startling conclusion that truth is made by men, and that mathematics is created by mathematicians, and that Columbus created America; but common sense, it is refreshing to think, is at any rate above being flattered by philosophical persuasion that it really occupies a place sometimes reserved for an even more Sacred Being."

Doubtless if Columbus were to discover America over again he might conclude that acts of creation had gone on in the meantime, and might reasonably assume that they happened in the past, and doubtless Mr. Jourdain is forced to conclude from his own argument that the words he uses in the English tongue have not been built up by the efforts of man but have existed from the beginnings of time, that the idea of propositional function and of relative and of function, pointset, transfinite number, Lobatchevskian space, and a long list of other terms, have always been waiting in the mines of thought for the lucky prospector, but common sense would refute this view with very little study of the case. We may grant that electric waves have always existed, but that the wireless telegraph has always existed in any sense is not true; nor that even if carbon, nitrogen, hydrogen and oxygen have always existed, nitroglycerine

is to be dug out of wells, or that because sound-waves exist in the air, that therefore symphonies, operas, and all music have always been waiting to be discovered, not created. It is true perhaps that the elementary units out of which things material or mental are constructed exist in some sense, external to any one individual in some sense, but it is not true that therefore the combinations of these elements have always existed. Logistic, with all its boasted power, has never constructed a theorem that was truly synthetic in character, it has never taken a set of new postulates not derived from previously existing theories and developed a branch of mathematics similar to geometry or an algebra. It is powerless to move without the constant attendance of the intellect, it draws no more conclusions than Jevons's logical machine without its operator. It has never even introduced as one of its results a new thought of wide-reaching power, such as the idea of propositional function itself. This idea came from the extension of the mathematical function to other things than quantity. Columbus did not create the trees nor Indians nor shores of America, but he did create something that the Icelanders or the Chinese or other reputed previous discoverers did not create, and its existence we celebrate to-day more than the forgotten Indians, or the shifting sands of Watling's Island, or the broken tree-trunks. Mathematics, as we said before, did not spring like Athena from the head of Zeus, nor is it the record of the intellectual microscope and scalpel, but rather as Pringsheim,[10] who is not a philosopher but a mathematician, says: "The true mathematician is always a good deal of an artist, an architect, yes, a poet. Beyond the real world, though perceptibly connected with it, mathematicians have created an ideal world which they attempt to develop into the most perfect of all worlds, and which is being explored in every direction. None has the faintest conception of this world

[10] *Jahr. Deut. Math. Ver.*, Vol. XXXII, p. 381.

except him who knows it; only presumptuous ignorance can assert that the mathematician moves in a narrow circle. The truth which he seeks is, to be sure, broadly considered, neither more nor less than consistency; but does not his mastership show indeed in this very limitation? To solve questions of this kind he passes unenviously over others."

We must pass on however to the reef that wrecked logistic in its short voyage after imperial dominion. This is nothing less than infinity itself. Since logistic asserted philosophically the suprasensible and supramental existence of its objects, it was forced to assert that there is an absolute infinity. In the transfinites of Cantor it found ultimately its ruin. In order to handle classes that had an infinite number of members it had to set up definitions that ultimately led to the contradictions which in the *Principles of Mathematics* of Russell were left unsolved. These were the objects of the assaults of Poincaré and others, and led to the definitive abandonment of the second volume of the *Principles*. The presentation of the *Principia* has many modifications, too long to cite, but the discussions in the *Revue de métaphysique et de morale* from 1900 on will be found very illuminating in their bearing on the nature of mathematics. The philosophical writings of Poincaré particularly should be consulted. The net result of all the discussions is that all the metaphysics has been eliminated from logistic, and it assumes its proper place in the mathematical family, as a branch of mathematics on a par with the other branches, such as arithmetic, geometry, algebra, group-theory.

The question of infinity is one of the most difficult to consider, and in one of his last articles Poincaré despairs of mathematicians ever agreeing upon it. The reason for perpetual disagreement he gives is the fundamental difference in point of view of reasoning in general. If the objects of mathematics are supramental, then the mind is forced

to admit an absolute infinity. If the objects of mathematics are created by the mind, then we must deny the absolute infinity. So far no decisive criterion has appeared, beyond that laid down by Poincaré, that any object about which we talk or reason must be defined, that is, made to be distinguishable from all other objects, in a finite number of words. For example, there is no such thing as the collection of all integers, since while we may define the class of integers and also any one integer we cannot define each and every integer. When logistic seeks to correlate the collection of all integers to any other infinite collection, member to member, this criterion demands that a law of correlation be stated which may be applied to every member of the class. This is manifestly impossible. A case is the proof that rational numbers may be put into a one-to-one correspondence with the integers. While any one rational may be placed in this way, or any finite number of them, yet according to the criterion it is not possible to decide that we can place every rational in this way. Manifestly any operation that has to be done in successive steps will never reach an absolute infinity. All proofs relating to infinite collections consider that the statement of a law for any member of the class is sufficient. The criterion demands a law for every member, which is admittedly not possible. The absolute infinity must not be confused with the mathematical infinity, which is merely an unlimited or arbitrary class. In all the processes we use in getting limits, the infinity that enters is not the Cantor infinity.

We may then safely conclude that logistic furnishes truth to the other branches of mathematics in much the same way that algebra does to geometry, or geometry to algebra, or numbers to group-theory, or hypernumbers to geometry. By logistic we may draw conclusions about the elements with which we deal. If we try to interpret the

conclusions logistic is powerless to do so any more than geometry can yield us theorems in logic. Also the processes in reasoning of any kind are no different in logistic from what they are in algebra, geometry, theory of numbers, theory of groups, and it is the intelligence, not the logistic, that draws the conclusion of logistic, just as it is the mathematician that solves algebraic equations, not algebra. Logistic has a right therefore to exist as an independent branch of mathematics, but it is not the Overlord of the mathematical world. As to the philosophical import of logistic, we may well follow Poincaré's advice, and continue the development of mathematics with little concern whether realism or idealism or positivism is substantiated in the philosophical world. Indeed we may conclude eventually with Lord Kelvin[11] that "mathematics is the only true metaphysics."

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[11] *Life*, p. 10.

RICHARD DEDEKIND.

(1833-1916.)

ON February 12, 1916, Julius Wilhelm Richard Dedekind died at his native Brunswick in Germany. He was one of the world's most distinguished workers at the theory of numbers, and in particular with Ernst Eduard Kummer and Leopold Kronecker at the theory of algebraic numbers; and most of his work is described in supplements to his editions of Dirichlet's *Vorlesungen über Zahlentheorie*.[1] In these supplements we can find references to his fundamental and enormously important ideas on the nature and meaning of numbers.

From the point of view of the fundamental principles of mathematics and the closely allied questions of logic and philosophy, the most important works of Dedekind are on the explanation of "continuity" by comparison with the system of real numbers, in which the irrational numbers were defined in a memorable way, and on the exceedingly subtle question of the definition, by logical concepts alone, of the integer numbers. Both of Dedekind's classical pamphlets: *Stetigkeit und irrationale Zahlen* of 1872 and *Was sind und was sollen die Zahlen?* of 1888 have been translated into English by W. W. Beman under the title: *Essays on the Theory of Numbers: I. Continuity and Irrational*

[1] A short indication of Dedekind's mathematical works was given by G. B. Mathews in *Nature*, Vol. XCVII, 1916, pp. 103-104.

Numbers; II. The Nature and Meaning of Numbers.[2] It is to this translation that the notes below refer.

The ideas of Dedekind on the nature and meaning of numbers, which are here described (§ II) after his logically subsequent and historically earlier work on continuity (§ I), led Dedekind to work out—apparently in complete independence of the previous work of De Morgan and the contemporary work of Charles Peirce—the greater part of what is now known as "the logic of relations." On another occasion I hope to give an account of later critical and constructive work on both these contributions of Dedekind to the principles of mathematics.

I.

In the autumn of 1858, Dedekind, who was then professor at the Polytechnic School of Zurich, had, for the first time in his life, to lecture on the elements of the differential calculus, and then felt more acutely than ever before the lack of a really scientific foundation of arithmetic. "In discussing," he said, "the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually but not beyond all limits must certainly approach a limiting value, I had recourse to geometrical evidences. Even now I maintain that such an employment of geometrical intuition is, from a didactic standpoint, extraordinarily useful and indeed indispensable, if we do not wish to lose too much time. But no one will deny that this manner of introduction to the differential calculus can make no claim to scientific accuracy. In my own case this feeling of dissatisfaction was so overpowering that I made a firm resolve to meditate until I should find a purely arithmetical and completely rigorous foundation for the principles of infinitesimal analysis.

[2] Chicago and London: The Open Court Publishing Co., 1901.

People say so often that the differential calculus is occupied with continuous magnitudes, and yet nowhere is there given an explanation of this continuity; and even the most rigorous expositions of the differential calculus do not found their proofs on continuity, but appeal with more or less consciousness of the fact to geometrical notions or notions suggested by geometry, or rest on theorems which have never been proved arithmetically. To these belongs, for example, the above mentioned theorem, and a closer investigation convinced me that this or any equivalent theorem can be regarded, in a sense, as a sufficient foundation for infinitesimal analysis. So all reduced to the discovery of its real origin in the elements of arithmetic and thus to obtain at the same time an actual definition of the essence of continuity. I succeeded in doing this on November 24, 1858." Although Dedekind communicated his ideas and discussed them with some of his colleagues and pupils, he could not make up his mind for many years to let them be printed because "the exposition is not quite easy, and besides the matter itself is so unfruitful."[3] However, he had half determined to select that theme for a publication to be dedicated to his father on the celebration in April, 1872, of the fiftieth anniversary of his father's entry into office, when, in March of that year, he came across Heine's memoir in Vol. LXXIV of Crede's *Journal*, with which in essentials Dedekind agreed, "as indeed cannot be otherwise," but the form of his own work appeared to him to be simpler and to emphasize more precisely the main point. Also Dedekind remarked the identity of his axiom of the continuity of the straight line with Cantor's axiom, of which he read when writing his preface, and that he could not recognize the utility of Cantor's distinction of real numbers of still higher kind, because of his conception of the real number domain as complete in itself.

[3] *Stetigkeit*, (2d ed., 1892), p. 2; cf. *Essays*, p. 2.

Comparing the system of rational numbers, in order of magnitude, with the points of a straight line L, we see that, if any origin be taken on L and a fixed unit of measurement, to any rational number *a* can be constructed a corresponding point; but there are points (those determined by incommensurable lengths measured from o) to which no rational numbers correspond. Thus we can say that "L is infinitely richer in point-individuals than the domain R of rational numbers in number-individuals."[4] So if, as we wish,[5] all phenomena in the straight line are also to be followed out arithmetically[6] R must be refined by the creation of new numbers, and the domain of numbers raised to the same completeness—or "continuity"—as the straight line.

"The way in which irrational numbers are usually introduced is connected with the concept of extensive magnitude—which itself is nowhere rigorously defined—and explains number as the result of the measurement of one such magnitude by another of the same kind.[7] Instead of this I demand that arithmetic shall be developed out of itself. That such connections with non-arithmetical notions have furnished the immediate occasion for the extension of the number-concept may, in general, be granted (though this was certainly not the case in the introduction of complex numbers); but this surely is no sufficient ground for introducing these foreign connections into arithmetic, the science of numbers. Just as negative and fractional rational numbers must and can be formed by a new creation, and as the laws of operation with these numbers must and can be reduced to the laws of operation with posi-

[4] *Stetigkeit*, p. 9; *Essays*, p. 9. [5] "Was doch der Wunsch ist," *ibid.*
[6] Cf. *Stetigkeit*, pp. 5-6, 10; *Essays*, pp. 4, 10.
[7] "The apparent advantage of this definition of number in point of generality vanishes the moment we think of complex numbers. In my view, the conception of the ratio to one another of two magnitudes of the same kind can be clearly developed only after the irrational numbers have been introduced."

tive integers, so we must endeavor completely to define irrational numbers by means of the rational numbers alone. There only remains the question as to how to do this."[8]

Now the essence of this "continuity" of L was found by Dedekind[9] after long meditation to be: If all the points of L fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which generates this division. This, as Dedekind emphasized, will probably be considered as evidently true by every one; it cannot be proved, but is an axiom by means of which we first recognize the line of its continuity. If space has a real existence, it need not necessarily be continuous; many of its properties would remain the same if it were discontinuous[10]; and if we knew that it was discontinuous, nothing could prevent us, if we wished, making it continuous in thought by filling up its lacunae. Another simple logical transformation of the above axiom is not so obvious: there is one and only one point (of the first class) which is on the extreme left of the first class, or one and only one of the second class on the extreme right of the second class, but not both.

[8] *Steigkeit*, p. 10; *Essays*, pp. 9-10.

[9] *Stetigkeit*, p. 11; *Essays*, p. 11. This axiom has been frequently misunderstood; thus L. Couturat (*De l'infini mathématique*, Paris, 1896, p. 416) stated it: "If all the quantities of a kind can be divided into two classes such that all the quantities of the one precede (or follow) all those of the other, there exists a quantity of this kind which *both* follows all those of the inferior class *and* precedes all those of the superior class." Russell, in a review (*Mind*, Vol. VI, 1897, p. 117), rightly pointed out the mistake in this wording but wrongly advanced the same criticism against Dedekind's own axiom (*The Principles of Mathematics*, Vol. I, Cambridge, 1903, p. 279). In fact, we do not need, as Russell presumed, a "point left over to represent the section"; and Russell's (second) "emendation" (pp. 279-280) is Dedekind's original axiom.

[10] An example of this was given in the preface of *Was sind und was sollen die Zahlen?* (*Essays*, pp. 37-38). Choose any three points A, B, C, which do not lie in a straight line and which are such that the ratios of their distances AB, AC, BC are algebraic numbers; and regard as present in space only those points M for which the ratios of AM, BM, and CM to AB are algebraic numbers. The space consisting of the points M is everywhere discontinuous, but yet in it all the constructions in Euclid's *Elements* can be carried out just as well as in a continuous space.

The purely arithmetical definition of new numbers among those of the system R so as to make it a continuous system was now brought about on a basis analogous to that of the above axiom. Any rational number a brings about a division of the system R into two classes A_1, A_2, such that any number of A_1 is smaller than any number of A_2; a is either the greatest of A_1 or the least of A_2. If now we have any division of R into classes A_1, A_2, such that any member of A_1 is smaller than any member of A_2, we call such a division a "section" or "cut" (*Schnitt, coupure*), and denote it by (A_1, A_2). We can then say that any rational number a generates a section, or strictly speaking two sections (which, however, we will not regard as essentially different).[11] But there are an infinity of sections—such as that where A_1 consists of all the rational numbers r such that $r^2 < D$ is a positive non-square integer, and A_2 of the rest—which are not generated by rational numbers,—that is to say, neither has A_1 a maximum nor A_2 a minimum; and in this consists the incompleteness or discontinuity of R. Now, whenever we have a section (A_1, A_2) generated by no rational number, we create (*erschaffen*) a new, an "irrational number," which we regard as completely defined by the section (A_1, A_2) and is said to generate it.[12]

By comparing two sections, (A_1, A_2) and (B_1, B_2), as to the inclusion or not of any term of A_1 in B_1, or *vice versa*, we arrive at a basis for determining the order of any two real (rational or irrational) numbers α and β as symbolized by

$$\alpha = \beta,\ \alpha > \beta,\ \text{or}\ \alpha < \beta;$$[13]

and also definitions of new sections whose generators may be represented by

[11] *Stetigkeit*, p. 12; cf. *Essays*, p. 13.
[12] *Stetigkeit*, p. 14; *Essays*, p. 15.
[13] *Stetigkeit*, pp. 15-19; *Essays*, pp. 15-21.

$$\alpha+\beta,\ \alpha-\beta,\ \alpha.\beta \text{ and } \alpha^{\beta},$$

may be given.[14]

We will now indicate the use of the-conception of a section to prove the theorems on limits mentioned above.[15] A variable x is said to have a fixed limiting value α, when $x-\alpha$ ultimately sinks, numerically speaking, below any positive, non-zero, number; and our first theorem is that, if x increases continually, but not beyond all values, it approaches a *definite* limit. By the supposition, we have numbers α_2 such that we always have $x<\alpha_2$; denote the totality of these numbers by A_2, and that of the other real numbers by A_1. Any member (α_1) of A_1 has the property that in course of the process $x\leqq\alpha_1$, and so every member of A_1, is smaller than any member of A_2, so that (A_1, A_2) is a section. Its generator (α) is either the greatest in A_1 or the least in A_2; the former cannot be the case, because x never ceases to increase. Thus α is the least member of A_2, and it is a limit of the x's, for, whatever member of A_1 the number α_1 may be, we ultimately have $\alpha_1<x<\alpha$.

Still more often used is the equivalent of this theorem: If, in the process of variation of x, for any positive δ (however small) a corresponding place can be given from which one x varies by less than δ, then x approaches a limiting value. This can easily be derived from the foregoing theorem, or directly, as we do here, from the principle of continuity.

If $x=a$ at the instant referred to in the theorem, ever afterwards $x>a-\delta$ and $x<a+\delta$. On this fact we found a double separation of the system of real numbers. Put every number α_2 such that, in the course of the process, we have $x\leqq\alpha_2$, in a class A_2, and let A_1 consist of all the other numbers; so that, if α_1 is such a number it will happen

[14] *Stetigkeit*, pp. 19-22; *Essays*, pp. 21-24.
[15] *Stetigkeit*, pp. 22-24; *Essays*, pp. 24-27.

infinitely often, however far the process may have progressed, that $x > \alpha_1$. Since any α_1 is less than any α_2, there is a definite generator α of the section (A_1, A_2), which we will call the upper limiting value of x. Similarly, a second section (B_1, B_2) of the system of real numbers is brought about by x, if any number β_1 (such as $a - \delta$) such that in the course of the process $x \leqslant \beta_1$ is put in B_1; and the generator β is called the lower limit of x. The two numbers α and β are also evidently characterized by the property that, if ε is taken positive and arbitrarily small, we always have $x < \alpha + \varepsilon$ and $x > \beta - \varepsilon$, but never finally $x < \alpha - \varepsilon$ and never finally $x > \beta + \varepsilon$. Now, two cases are possible: if α and β are different from one another (so that $\alpha > \beta$), x oscillates, and suffers, however far the process may have progressed, variations whose amount exceeds $(\alpha - \beta) - 2\varepsilon$. But the original supposition, which is now first used, excludes this, and so there only remains the case $\alpha = \beta$; and we see that x approaches the limiting value α.

Dedekind remarked[16] that, while the lengthiness in the definitions of the elementary operations can partly be overcome by the use of auxiliary concepts such as that of an "interval" (a system of rational numbers such that, if a and a' are any members of it, all the numbers between a and a' are also members of it)[17] and of its limits, yet "still lengthier considerations seem to loom up when we wish to transfer the innumerable theorems of the arithmetic of rational numbers, as, for example, the theorem $(a + b)c = ac + bc$, to any real numbers. However, this is not so, for we soon convince ourselves that here all reduces to proving that the arithmetical operations themselves have a certain continuity. What I mean by this I will put in form of a general theorem: If the number λ is the result of a calcula-

[16] *Stetigkeit*, pp. 20-22; *Essays*, pp. 22-24.

[17] Both the classes of any section are "intervals."

tion undertaken with the numbers $\alpha, \beta, \gamma, \ldots$, and if λ lies inside the interval L, then intervals A, B, C,..., inside which $\alpha, \beta, \gamma, \ldots$, respectively lie, can be given such that the result of the same calculation in which $\alpha, \beta, \gamma, \ldots$ are replaced by any numbers of A, B, C,... respectively, is always a number lying inside L. The forbidding clumsiness, however, which marks the enunciation of such a theorem convinces us that here something must be done to aid language. This is done in the most satisfactory way by introducing the concepts of *variable magnitudes, functions,* and *limiting values*; and indeed the most convenient thing is to base the definitions of the simplest arithmetical operations on these concepts, but this cannot be carried farther here."[18]

II.

The last few words contain an indication of the fundamental concepts upon which Dedekind's theory of integers was based. The notion of an aggregate or "system" of things is, of course, the most fundamental, and also we utilize, in counting, the capability of the mind to *refer* things to things, to let a thing *correspond* to a thing, or to *image* (*abzubilden*) a thing by a thing. Without this capability no thought is possible, and on this single, but quite indispensable, foundation must, in Dedekind's view, the whole science of numbers be erected.[19] This idea of

[18] *Stetigkeit,* pp. 21-22; cf. *Essays,* pp. 23-24.

[19] In the eleventh appendix of Dedekind's edition of Dirichlet's *Vorlesungen über Zahlentheorie* (3d ed., 1879, § 163, p. 470), Dedekind said: "It happens very frequently in mathematics and other sciences that, if we have a system Ω of things or elements ω, every definite element ω is replaced according to a certain law by a definite element ω' corresponding to it. We are accustomed to call such an act a substitution and say that by this substitution the element ω passes over into the element ω' and the system Ω into a system Ω' of elements ω'. The expression of this is somewhat more convenient if we ....conceive this substitution as a transformation (*Abbildung*) of the system Ω." To this he added the note: "On this ability (*Fähigkeit*) of mind to compare a thing ω with a thing ω', or to refer ω to ω', or to let ω correspond to ω', without which thought is impossible, rests, as I will try to prove in another place, the whole science of numbers."

correspondence is the idea of functionality or, in other words, of establishing a *relation* between things.

Dedekind's views on the nature of numbers may be expressed as follows. Arithmetic, including Algebra and Analysis, "is a part of logic, and the number-concept is quite independent of the notions or intuitions of space and time, and is an immediate consequence of the pure laws of thought." Toward the beginning of his *Stetigkeit*,[20] he wrote: "I regard the whole of arithmetic as a necessary or at least natural consequence of the simplest arithmetical act, that of counting, and counting itself is nothing else than the successive creation of the infinite series of positive integers, in which each individual is defined by the one immediately preceding; the simplest act is the passing from an already formed individual to the consecutive new one to be formed. The chain of these numbers forms even by itself an exceedingly useful instrument for the human mind; it presents an inexhaustible wealth of remarkable laws obtained by the introduction of the four fundamental operations of arithmetic. Addition is the combination of any repetitions we wish of the above mentioned simplest act into a single act; from it in a similar way arises multiplication. While the performance of these two operations is always possible, that of the inverse operations, subtraction and division, proves to be limited. Whatever the immediate occasion may have been and whatever comparisons or analogies with experience or intuition may have led us, it is certainly true that just this limitation in performing the indirect operations has in each case been the real motive for a new creative act. Thus negative and fractional numbers have been created by the human mind; and in the system of all rational numbers there has been gained an instrument of infinitely greater perfection. Numbers are free creations of the human mind; they serve as a

[20] Pp. 5-6; cf. *Essays*, p. 4.

means to grasp the difference of things more easily and distinctly. Only by means of the purely logical structure of the science of numbers and the continuous number-region obtained in it are we in a position accurately to investigate our notions of space and time, by referring them to this number-domain created in our mind."

Dedekind had the intention of showing the development of the conception of the natural (integral) numbers from the purely logical conceptions of aggregate and "representation" (*Abbildung*), before the publication (1872) of his work on continuity, but it was only after the appearance of this work that, from 1872 to 1878, he wrote out a sketch of his system containing all its esssential ideas, and showed it to and discussed it with many mathematicians. In 1887 a careful exposition was carried out and published in the next year under the title *Was sind und was sollen die Zahlen?*[21] The motive for the publication was the appearance of the essays of Kronecker and von Helmholtz. His own work, as he said, though similar in many respects to those essays, was in its foundations essentially different, and he had formed his own view "many years before and without influence from any side."

Dedekind regarded the maxim that "in science anything which can be proved is not to be accepted without proof"[22] as unfulfilled even in the most recent methods of laying the foundations of arithmetic. And Dedekind's answer to this want was one of the first examples of that tendency of modern mathematics to extend exactness of treatment to the very principles, that has been gradually carried out by mathematical logicians like Frege, Peano and Russell.

As we should expect, the tract at first excited the derision of those unperceiving mathematicians who thought

[21] Brunswick, 1888; second unaltered edition, 1893 [prefaces dated Oct. 5, 1887 and Aug. 24, 1893]; *Essays*, pp. 31-115.

[22] *Was sind und was sollen die Zahlen?*, p. vii; cf. *Essays*, p. 31.

that Dedekind was merely taking an unnecessarily long time to prove obvious things like the commutative law in arithmetic. That such things seem to be immediately obvious will at once be granted, but the logical problem which interested Dedekind and many others since about the middle of the nineteenth century was whether or not such theorems are logically implied by those (logical) principles which hold for all true thought without exception, and are not of merely empirical validity. If we are in sympathy with efforts to solve the problems of the nature of our knowledge, we ought not to complain that the detailed writing out of logical steps takes up a large space. Besides, such a complaint is irrelevant.

Dedekind considered what he called "systems," which are what logicians call "classes" and mathematicians now usually call "aggregates," and then the idea of a correspondence of the elements of a system with elements of another system or with one another. He viewed such a correspondence as a "transformation"; and, when he came to consider "similar [or one-to-one] transformations of a system into a part of itself," he arrived at defining an "infinite" system[23] and thus fell upon much the same ideas that Georg Cantor independently did.[24] A special infinite system is the "simply infinite system" N which is such that there exists a similar transformation φ such that $\varphi(N)$ is a part of N, and N is the common part of all systems S which contain a definite element of N which is not of $\varphi(N)$, and for which $\varphi(S)$ is a part of S.[25] We can see without much difficulty that N consists of an element a, its transform a', the transform a'' of a', and so on; but it is to be noticed that Dedekind defines his infinite systems as wholes and does not use the vague words "and so on."

[23] *Essays*, pp. 63, 41-42.

[24] *Contributions to the Founding of the Theory of Transfinite Numbers*, Chicago and London, 1915, p. 41.

[25] Cf. *Essays*, pp. 67, 56-58.

The ordinal numbers then appear as mental abstractions from such systems as N,[26] the theorem of complete induction is proved for them,[27] and the various other fundamental arithmetical concepts and theorems established. In particular, Dedekind considered cardinal numbers to be logically subsequent to ordinal numbers.[28]

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[26] *Ibid.*, p. 68.
[27] *Ibid.*, pp. 69-70, 60-62, 32-33, 42-43.
[28] *Ibid.*, pp. 109-110, 32.

CRITICISMS AND DISCUSSIONS.

THE ARITHMETICAL PYRAMID OF MANY DIMENSIONS

AN EXTENSION OF PASCAL'S ARITHMETICAL TRIANGLE TO THREE AND MORE DIMENSIONS, AND ITS APPLICATION TO COMBINATIONS OF MANY VARIATIONS.

I.

In 1665 Pascal wrote his *Traité du triangle arithmétique* and showed that the system of numbers there developed, the so-called *figurate* numbers, had many remarkable properties. The most useful of these, and for our present purposes the most important, is the fact that this table gives the value of the expression ${}_nC_r$, for all positive integral values of n and r (including 0). The expression ${}_nC_r$ means the number of combinations of n things taken r at a time. It is also written $\binom{n}{r}$, and is equal to

$$\frac{n(n-1)(n-2)(n-3)\ldots\ldots(n-r+1)}{r!}$$

or $n!/r!\,(n-r)!$, in which $r!$ is read "factorial r" and denotes the product of all the integral numbers from 1 to r inclusive. The appropriate solution for any given values of n and r is to be found in the nth line and rth column of the arithmetical triangle. See Table I.

Now ${}_nC_r$ refers to things, each of which is capable of two and only two variations, such as coins that may fall either heads or tails. But frequently we have to do with things subject to more than two variations, such as a number of signal lights each showing several colors, or a number of dice which may fall on any one of their six faces. The solutions of such cases are not to be found in the arithmetical triangle, though in every case they can be shown to be

Names of the Columns			Natural Nos.	Tri-angular Nos.	Py-ramidal Nos.	Pentahe-droidal Nos.	Hexahe-droidal Nos.	Septahe-droidal Nos.	Octahe-droidal Nos.	Nonahe-droidal Nos.	Dekahe-droidal Nos.	Undeka-hedroi-dal Nos.	
Boundaries of the Figures			Corners	Edges	Sur-faces	Tetra-hedra	Pentahe-droids	Hexahe-droids	Septahe-droids	Octahe-droids	Nonahe-droids	Dekahe-droids	
		0	1	2	3	4	5	6	7	8	9	10	Sum
	0	1											1
Point	1	1	1										2
Line	2	1	2	1									4
Triangle	3	1	3	3	1								8
Tetrahedron	4	1	4	6	4	1							16
Pentahedroid	5	1	5	10	10	5	1						32
Hexahedroid	6	1	6	15	20	15	6	1					64
Septahedroid	7	1	7	21	35	35	21	7	1				128
Octahedroid	8	1	8	28	56	70	56	28	8	1			256
Nonahedroid	9	1	9	36	84	126	126	84	36	9	1		512
Dekahedroid	10	1	10	45	120	210	252	210	120	45	10	1	1024
Combinations		${}_nC_0$	${}_nC_1$	${}_nC_2$	${}_nC_3$		${}_nC_r$			${}_nC_{n-2}$	${}_nC_{n-1}$	${}_nC_n$	
General Formulas		$\frac{n!}{0!\,(n-0)!}$	$\frac{n!}{1!\,(n-1)!}$	$\frac{n!}{2!\,(n-2)!}$	$\frac{n!}{3!\,(n-3)!}$		$\frac{n!}{r!\,(n-r)!}$			$\frac{n!}{(n-2)!\,2!}$	$\frac{n!}{(n-1)!\,1!}$	$\frac{n!}{n!\,0!}$	2^n

TABLE I. The Arithmetical Triangle.
See also Table VI for another arrangement.

the product of two or more numbers there to be found. So far as the writer is aware no systematic method of selecting the proper factors has yet been given.

In the case of two variations, for any given value of n there will be $n+1$ classes, obtained by giving r successively all integral values from 0 to n. In any class r is the number of one kind present, $n-r$ the number of the other. These can all appropriately be arranged along a straight line. In fact the complete set of solutions is to be found in the nth line of the arithmetical triangle. But if the n things are capable of more than two variations—if for example they may be A's, B's, C's, D's, etc.—then a much larger number of classes arises; for to any one of these letters may be assigned in turn all the integers from 0 to n, and all vary independently. These classes cannot be so simply arranged, and the task of obtaining all of them and calculating the number of combinations for each becomes very complicated. Some systematic method must be adopted to insure exhaustive enumeration.

The object of the present paper is to show how these cases of many variations may be appropriately arranged in more-dimensional tables, so as to develop with certainty all possible classes, and show their proper relations to one another, and also to show how the arithmetical triangle may likewise be extended to more dimensions, and thus provide means of readily finding the number of combinations corresponding to each class. The method is somewhat complicated to explain, but easy to operate. We shall begin by describing a few of the many remarkable properties of the arithmetical triangle, such as will be useful to us, and then take up in turn its extension to 2, 3, 4,. . . . , k variations.

All the numbers of the arithmetical triangle can of course be calculated from the general formula already given, $n!/r!\,(n-r)!$. But the table can also be much more simply produced by a process of successive addition as follows: Beginning with 1, below any line write the same line moved one place to the right and add. The result is the next line. The process is shown in Table II.

From the mode of development it is apparent that the differences of any column are to be found in the next column to the left. Any column is therefore an arithmetical series of the rth order, whose rth differences are constant and equal to 1. The table is in fact the complete system of all arithmetical series whose final differences are 1. Conversely each number gives the sum of all the

preceding numbers of the next column to the left, or the sum of any two numbers in the same line is found immediately below the right-hand one.

Each line gives the binomial coefficients in order for the exponent corresponding to the number of the line, for these coefficients are also given by the formula $\binom{n}{r}$. The sum of all the numbers of any line is 2^n.

The columns have been given special names because of certain properties they possess. The zero column is composed only of units. The first column contains the natural numbers. The second contains the triangular numbers, so called because they give the number of units that can be arranged in a triangle, having succes-

Line Zero	1
	1
Line One	1 1
	1 1
Line Two	1 2 1
	1 2 1
Line Three	1 3 3 1
	1 3 3 1
Line Four	1 4 6 4 1

TABLE II. Method of Constructing Arithmetical Triangle.

sively 1, 2, 3, 4,... units on a side. The third column contains the pyramidal numbers, so called because they give the number of units that can be piled like cannon balls in the form of a triangular pyramid or tetrahedron, having successively 1, 2, 3, 4,... units on a side.

The remaining columns have as yet received no special names, but they might appropriately be named after the succeeding higher-dimensional pyramids, since they similarly give in turn the numbers of units that can be arranged in the form of these higher pyramids, having successively 1, 2, 3, 4,... units on a side. We shall call the latter, after Stringham, successively, the 4-dimensional pentahedroid, the 5-dimensional hexahedroid, the 6-dimensional septahedroid, etc., in general the $(k-1)$-dimensional k-hedroid, and name the columns

after them as shown in Table I. Thus the tetrahedron becomes a 4-hedroid, the triangle a 3-hedroid, the line a 2-hedroid, the point a 1-hedroid, and the corresponding columns the 4, 3, 2, 1-hedroidal numbers respectively.

Most useful for our subsequent purposes however is the fact that the arithmetical triangle gives complete specifications for the construction of any of these higher pyramids. Thus the nth line gives in order the number of 0, 1, 2, 3,.... $(n-1)$-dimensional boundaries of the $(n-1)$-dimensional n-hedroid. We have only to read the line designating the succeeding numbers in turn as so many corners, edges, surfaces, tetrahedra, etc., as indicated in Table I. Thus we may read:

First line: One point has 1 corner or 0-space boundary.

Second line: One line has 2 corners or ends, and 1 edge or interior 1-space.

Third line: One triangle has 3 corners, 3 edges, and 1 interior 2-space or surface.

Fourth line: One tetrahedron has 4 corners, 6 edges, 4 surfaces, and 1 interior 3-space.

Fifth line: One pentahedroid has 5 corners, 10 edges, 10 triangular surfaces, and 1 interior space of four dimensions.

Similarly the remaining lines may be read.

It may be noted that as the line lies between its ends, the triangle within its edges, the 3-space of the tetrahedron inside its bounding 2-space surfaces, so the 4-space of the pentahedroid is *inside* its bounding 3-space tetrahedra. Similarly with the higher pyramids. The 5-space is inside the 4-space, the 6- inside the 5-, etc. We get to higher and higher degrees of insideness.

As we shall use these higher pyramids to represent our combinations of many variations, it is important to know how they are constructed.

We may now proceed to our task of applying the arithmetical triangle to the cases of two and more variations. Calling k the number of variations, it will be found in every case that a $(k-1)$-dimensional k-hedroidal table will be required, the total number of classes is given by the $(n+1)$th k-hedroidal number, while the sum of all the combinations is k^n. This gives a valuable check on the correctness of the work. The variations we shall call A, B, C, D, etc.

Let $k=2$. The complete set of solutions for this case, as has already been stated, is to be found in the nth line of the arithmetical

triangle. They form therefore a linear table, as shown for $n = 10$, in Table III. The second half of any such line is always the reverse of the first half, so that there are only $(n+2)/2$, disregarding remainder, different numbers in it.

	0	1	2	3	4	5	6	7	8	9	10	B
A	10	9	8	7	6	5	4	3	2	1	0	
	1	10	45	120	210	252	210	120	45	10	1	

TABLE III. All Classes Combinations of 10 Things each susceptible of 2 Variations, such as 10 Coins, that may fall Heads or Tails.

Number of classes $= n + 1 = 11$. Sum of combinations $= 2^{10} = 1024$.
Note. First line shows number of B's present, second line number of A's.

Let $k = 3$. Each thing may now be an A or a B or a C. The additional classes possible may be derived from the previous case in the following way. To each class of A and B, as shown in Table III, additional classes may be formed, by keeping the B's constant, and exchanging the A's one by one for C's. The number of new classes thus formed will in each case be equal to the number of A's contained in the original cell. Thus the first cell gives 10, the second 9, the third 8, etc. new classes. The only appropriate way to arrange the new classes is in columns vertically under the cells from which they were developed, according to descending values of A, or ascending values of C. In this way the triangular Table IV is obtained, for the value of $n = 10$. It may be noted that there will be three classes where the n things are all of one kind, as all A's, all B's, or all C's. These are placed at the corners, or 0-dimensional boundaries of the triangle. There will be three sets of classes where the things are AB, BC, or AC. Each of these three sets is a duplicate of Table III, and they are located on the three edges of the triangle. The remaining classes in which all three variations appear are all located in the interior of the triangle, or in 2-space.

Further the triangle may be divided, as shown by the dash lines of Table IV, into a series of similar concentric triangular shells. The first interior shell contains all classes where one of the letters or variations appears each time only once. The second interior shell contains all classes where one of the letters appears always twice, the third three, etc. There will always be $n/3$, disregarding remainder, of these interior shells. It may further be noted that in each line,

column and diagonal, the second half is the reverse of the first half. The same numbers are hence oft repeated. In fact in Table IV there are only 14 different numbers, while if we deduct the 6 taken direct from Table III there are only 8 new numbers to calculate. These are shown enclosed in heavy lines in the table. It is easily seen that they cover approximately $\frac{1}{6}$ of the total area of the triangle, hence may be calculated for any n from $n^2/12$, taking the

A	0	1	2	3	4	5	6	7	8	9	10	B
0	1	10	45	120	210	252	210	120	45	10	1	
1	10	90	360	840	1260	1260	840	360	90	10		
2	45	360	1260	2520	3150	2520	1260	360	45			
3	120	840	2520	4200	4200	2520	840	120				
4	210	1260	3150	4200	3150	1260	210					
5	252	1260	2520	2520	1260	252						
6	210	840	1260	840	210							
7	120	360	360	120								
8	45	90	45									
9	10	10										
10	1											
C												

Note. The first line shows number of B's present, left-hand column the number of C's. The remaining $n-(B+C)$ are A's.

Number of classes = 11th triangular number or $[(n+1)(n+2)]/2 = 66$.

Sum of all combinations $= 3^{10} = 59{,}049$.

TABLE IV. Combinations of 3 Variations for $n = 10$.

nearest whole number. For the values of n from 1 to 15 the following are obtained as the new interior numbers to calculate:

$n =$ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15.
Nos. 0, 0, 1, 1, 2, 3, 4, 5, 7, 8, 10, 12, 14, 16, 19.

The fact that for the first two cases no new numbers are obtained means that all the cells abut on the edges. When $n = 3$ we get 1 interior cell; when $n = 4$ we get 3 cells but all are filled with the same number, viz. 12, as the reader may verify, if he likes, by working out these simple cases. Hence the number of calculations to make is not very great unless n is large.

It will be noted of course that the numbers increase in size toward the geometric center of the triangle. Also from the method

of numbering, the B's are constant in any column, the C's along any line, and each increases in value from the right angle outward. The A's are constant along any diagonal and increase in value toward the right angle, where they are all A's. This method of lettering and numbering will be adhered to in all that follows, and the characteristics that depend upon it will naturally always recur.

The value of any interior number is given by the general formula $\frac{n!}{A!B!C!} = \frac{n!}{B!C![n-(B+C)]!}$, since $A+B+C=n$. Calling $0!=1$, as is customary, this formula will also apply to the edges and hence to all the numbers of the triangle. But there is a much easier way of deriving the appropriate numbers directly from the arithmetical triangle, which it is one of the objects of this paper to show. If in the above formula we give to A, B and C different values, and write them down in their proper places, we obtain Table V, below. Since, as already stated, the A's, B's and C's are constant respectively along the diagonals, columns and lines it is apparent that the expression $n!/A!B!C!$ can, in three ways, be divided into two factors, one of which is constant and the other variable, according as we choose the constant part along a diagonal, a column or a line. Thus,—

$$\frac{n!}{B!C![n-(B+C)]!} = \frac{n!}{B!} \times \frac{1}{C![n-(B+C)]!} \quad (1)$$

$$\text{or} = \frac{n!}{C!} \times \frac{1}{B![n-(B+C)]!} \quad (2)$$

$$\text{or} = \frac{n!}{[n-(B+C)]!} \times \frac{1}{B!C!} \quad (3)$$

In every case the first factor is constant, the second variable. In (1) first factor is constant along a column, in (2) along a line, in (3) along a diagonal. Not much is gained by this, but if we multiply numerator and denominator in the three cases respectively by $(n-B)!$, $(n-C)!$, $(B+C)!$, we obtain,

$$\frac{n!}{B!C![n-(B+C)]!} = \frac{n!}{B!(n-B)!} \times \frac{(n-B)!}{C![n-(B+C)]!} = {}_nC_B \times {}_{n-B}C_C \quad (4)$$

$$\text{or} = \frac{n!}{C!(n-C)!} \times \frac{(n-C)!}{B![n-(B+C)]!} = {}_nC_C \times {}_{n-C}C_B \quad (5)$$

$$\text{or} = \frac{n!}{(B+C)![n-(B+C)]!} \times \frac{(B+C)!}{B!C!} = {}_nC_{B+C} \times {}_{B+C}C_B \quad (6)$$

In all three cases now each factor is seen to be a figurate number, hence one to be found in the arithmetical triangle. Moreover

A	0	1	2	3	4	5	6	B
0	$\frac{n!}{0!\,0!\,[n-(0+0)]!}$	$\frac{n!}{1!\,0!\,[n-(1+0)]!}$	$\frac{n!}{2!\,0!\,[n-(2+0)]!}$	$\frac{n!}{3!\,0!\,[n-(3+0)]!}$	$\frac{n!}{4!\,0!\,[n-(4\times 0)]!}$	$\frac{n!}{5!\,0!\,[n-(5+0)]!}$	$\frac{n!}{6!\,0!\,[n-(6+0)]!}$	
1	$\frac{n!}{0!\,1!\,[n-(0+1)]!}$	$\frac{n!}{1!\,1!\,[n-(1+1)]!}$	$\frac{n!}{2!\,1!\,[n-(2+1)]!}$	$\frac{n!}{3!\,1!\,[n-(3+1)]!}$	$\frac{n!}{4!\,1!\,[n-(4+1)]!}$	$\frac{n!}{5!\,1!\,[n-(5+1)]!}$		
2	$\frac{n!}{0!\,2!\,[n-(0+2)]!}$	$\frac{n!}{1!\,2!\,[n-(1+2)]!}$	$\frac{n!}{2!\,2!\,[n-(2+2)]!}$	$\frac{n!}{3!\,2!\,[n-(3+2)]!}$	$\frac{n!}{4!\,2!\,[n-(4+2)]!}$			
3	$\frac{n!}{0!\,3!\,[n-(0+3)]!}$	$\frac{n!}{1!\,3!\,[n-(1+3)]!}$	$\frac{n!}{2!\,3!\,[n-(2+3)]!}$	$\frac{n!}{3!\,3!\,[n-(3+3)]!}$				
4	$\frac{n!}{0!\,4!\,[n-(0+4)]!}$	$\frac{n!}{1!\,4!\,[n-(1+4)]!}$	$\frac{n!}{2!\,4!\,[n-(2+4)]!}$					
5	$\frac{n!}{0!\,5!\,[n-(0+5)]!}$	$\frac{n!}{1!\,5!\,[n-(1+5)]!}$						
6	$\frac{n!}{0!\,6!\,[n-(0+6)]!}$							
C								

TABLE V.

Formulas for Calculating Combinations of 3 Variations.

This table is of indefinite Extent, but for any given value of n comes to an end when $n = B + C$.

First factorial of denominator gives number of B's.
Second " " " " " " C's.
Third " " " " " " A's.

General formula: $\frac{n!}{B!\,C!\,[n-(B+C)]!}$

the first factor is not only constant but in (4) it is *equal* to the number that stands at the head (and foot) of the corresponding column, in (5) to the number that stands at either end of the corresponding line, in (6) to the number that stands at either end of the corresponding diagonal of the table to be calculated, while in each case the second factor for these terms becomes 1. By the first scheme then all the members of any column could be obtained by multiplying the first one by the successive values of the second factor of (4), obtained by giving B and C the proper values. Similarly by the second scheme all the terms of any line could be obtained from the first one, while by the third scheme all the terms of any diagonal from the end ones. Any one of the three schemes would produce the whole triangle, and it would seem natural to choose either of the first two. However because of the occurrence of $n-B$, and $n-C$, in these two schemes, making it necessary to assign a definite value to n before anything can be done, they do not lend themselves so readily to general treatment as the third scheme. We shall accordingly adopt the latter.

The proper values of the second factor, or coefficient as we shall call it, could of course be found in line B, or C, of column $B+C$ of the arithmetical triangle. But if we give different values to B and C and write the resulting numbers down in their proper places in the triangle, we obtain Table VI. It is at once seen that they follow a very regular order, being in fact nothing other than the arithmetical triangle itself, with each column pushed up to the top. This is the usual arrangement of the figurate numbers. The diagonals of the new table are the lines of the old.

The procedure of calculating a triangle then is as follows. As the first line write the nth line of the arithmetical triangle. Any interior number is then calculated by multiplying the number that stands at the end of its diagonal by the coefficient shown in Table VI. The process is shown in Table VII. Or the process may be described thus: Each successive line is derived from the first line by discarding each time one additional term and multiplying the remaining terms first by the natural, then by the triangular, then by the pyramidal, etc., numbers in order. Or we may sum the whole thing up in one general rule:

The mth *line of a "surface triangle" is derived from the* nth *line of the arithmetical triangle by discarding* $m-1$ *terms and multi-*

plying the remaining terms by the m-hedroidal numbers in order, beginning with the first one.

This rule applies to the outside edge as well as to the inner lines. The reason for calling the figure a *surface triangle* is because the surfaces of all our subsequent higher pyramids will be composed of such figures.

We have used for purposes of illustration $n = 10$. If we give to n other values we shall obtain similar triangles, smaller or larger according to the value of n. If we construct a series of these from $n = 0$ up, and pile them all up on top of each other with their A

A	0	1	2	3	4	5	6	7	8	9	10	11	12	B
0	1	1	1	1	1	1	1	1	1	1	1	1	1	
1	1	2	3	4	5	6	7	8	9	10	11	12	Natural Nos.	
2	1	3	6	10	15	21	28	36	45	55	66	Triangular Nos.		
3	1	4	10	20	35	56	84	120	165	220	Pyramidal Nos.			
4	1	5	15	35	70	126	210	330	495	Pentahedroidal Nos.				
5	1	6	21	56	126	252	462	792	Hexahedroidal Nos.					
6	1	7	28	84	210	462	924	Etc.						
7	1	8	36	120	330	792								
8	1	9	45	165	496									
9	1	10	55	220										
10	1	11	66											
11	1	12												
12	1													
C														

TABLE VI.

Coefficients for Calculating Combinations of 3 Variations or General Table of Figurate Numbers.

vertices coinciding, we shall obtain the *arithmetical pyramid* of three dimensions, as shown in Fig. 1. This has the form of a number of cubes piled in a corner of the room. It can of course be indefinitely extended. In it will be found all classes of combinations for $k = 3$, just as in the plane arithmetical triangle are to be found all classes for $k = 2$. Each surface will be an arithmetical of the old sort, all the new classes of three variations being found in the interior cells.

As the numbers ${}_nC_r$ are binomial coefficients in the expansion of $(a+b)^n$, so the numbers representing combinations of three

variations are the trinomial coefficients arising in the expansion of $(a+b+c)^n$. Thus Table IV enables us at once to write out the expansion of $(a+b+c)^{10}$. In like manner the numbers representing combinations of k variations are k-nomial coefficients arising in the expansion of the nth power of a polynomial of k terms.

Let $k=4$. Each thing may now be an A or a B or a C or a D. In order to develop and represent all possible classes in their proper relations to each other, use will now have to be made of a three-

A	0	1	2	3	4	5	6	7	8	9	10	B
0	1 ×1	10 ×1	45 ×1	120 ×1	210 ×1	252 ×1	210 ×1	120 ×1	45 ×1	10 ×1	1 ×1	
1	10 ×1	45 ×2	120 ×3	210 ×4	252 ×5	210 ×6	120 ×7	45 ×8	10 ×9	1 ×10		
2	45 ×1	120 ×3	210 ×6	252 ×10	210 ×15	120 ×21	45 ×28	10 ×36	1 ×45			
3	120 ×1	210 ×4	252 ×10	210 ×20	120 ×35	45 ×56	10 ×84	1 ×120				
4	210 ×1	252 ×5	210 ×15	120 ×35	45 ×70	10 ×126	1 ×210					
5	252 ×1	210 ×6	120 ×21	45 ×56	10 ×126	1 ×252						
6	210 ×1	120 ×7	45 ×28	10 ×84	1 ×210							
7	120 ×1	45 ×8	10 ×36	1 ×120								
8	45 ×1	10 ×9	1 ×45									
9	10 ×1	1 ×10										
10	1 ×1											
C												

TABLE VII.

Method of Calculating Combinations for $n = 10$, $k = 3$, from 10th Line of Arithmetical Triangle, and Coefficients of Table VI. Result is Table IV.

Only the 8 numbers enclosed in heavy lines need be calculated, all the others being taken direct or repetitions.

dimensional arrangement. For to every class containing A, B and C, as shown in Table IV, additional classes may now be formed by exchanging the A's successively for D's. The only proper place to put these new classes is to build them out in the third dimension from the original cells of Table IV from which they were developed. The resulting arrangement will be pyramidal in form because of the regularly decreasing value of A from the corner outward to the last diagonal. Such a pyramid, as we read in the fourth line of the arithmetical triangle, will be composed of 4 corners, 6 edges,

4 surfaces and 1 interior space. These will carry respectively the classes where 1, 2, 3 and 4 variations are present. There will be the following groups of such classes:

1 Variation,	A, B, C, D,	$= {}_4C_1, = 4$
2 Variations,	AB BC CD / AC BD / AD	$= {}_4C_2, = 6$
3 Variations,	ABC ABD ACD BCD	$= {}_4C_3, = 4$
4 Variations,	ABCD	$= {}_4C_4, = 1$

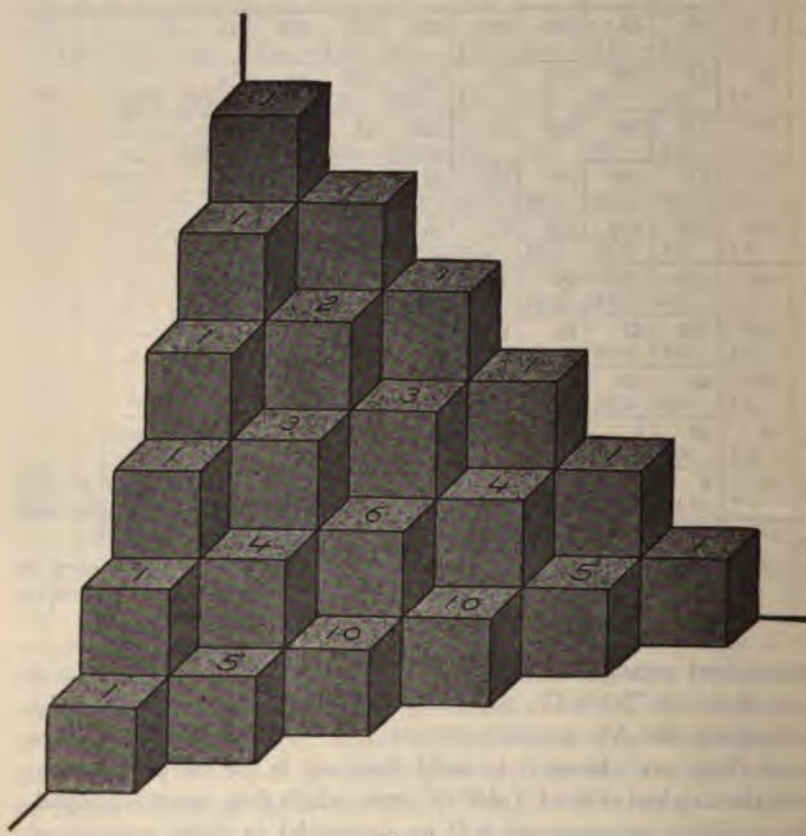

Fig. 1.

Each of the four surfaces of the pyramid will, for $n = 10$, be exactly the same as the triangle of Table IV, only as the 12 edges

of the 4 triangles coincide in pairs, thus reducing to 6, and as the 12 corners coincide in threes, thus reducing to 4, we cannot simply repeat the whole triangle four times, but regard must be had for the corners and edges to be omitted. One way would be to represent the corners, edges and interior part of the triangles separately. Another method is shown in Fig. 2. Here the slant sides are supposed to be folded down into the plane of the base, so as to depict all in one plane. The triangle BCD, which is in reality equilateral, is conveniently made right-angled like the others. The sides to be omitted are indicated by dotted lines. A modification of this plan, that economises space, is shown in Fig. 3. A modification of the first method is used in Diag. 1, Table VIII, and in most of the

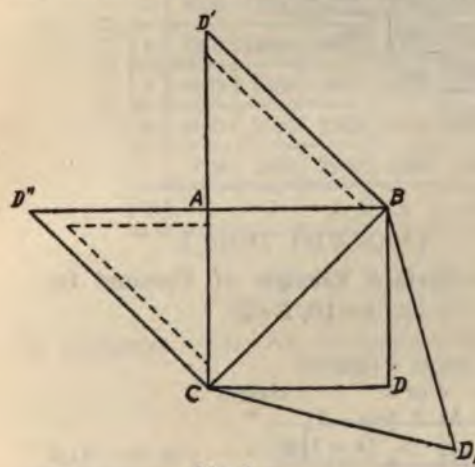

Fig. 2.

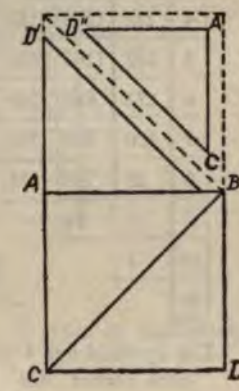

Fig. 3.

higher pyramids. The method of Fig. 2 is used in Diag. 5, Table XII, that of Fig. 3 in Diags. 2 and 3 of Table VIII.

The surface of the pyramid being thus represented, and no new numbers to calculate for it, it remains to consider only the representation and calculation of the interior cells, where four variations are present. To do this it will be found most convenient to consider the pyramid as made up of a number of concentric pyramidal shells, each one cell in thickness, like, to use an unsavory simile, the coatings of an onion. Each interior shell will then be exactly similar to the surface shell, which has just been pealed off. It can be represented in the same way, by any one of the three methods described. Analogous to the triangular shells of Table IV,

each cell of the first inner pyramidal shell will contain, beside the numbers indicated of the letters that stand at the vertices of the particular triangle in which it is found, one example of the missing letter. Thus the four triangles of the first inner shell are appropriately lettered, ABC+1D, ABD +1C, ACD+1B, DCB+1A, as in

A	0	1	2	3	4	5	6	7	8	9	10	B
0	1	10	45	120	210	252	210	120	45	10	1	
1	10	90	360	840	1260	1260	840	360	90	10		
2	45	360	1260	2520	3150	2520	1260	360	45			
3	120	840	2520	4200	4200	2520	840	120				
4	210	1260	3150	4200	3150	1260	210					
5	252	1260	2520	2520	1260	252						
6	210	840	1260	840	210							
7	120	360	360	120								
8	45	90	45									
9	10	10										
10	1											
C												

TABLE VIII, DIAG. 1.
Typical Surface Triangle of Pyramid for $n = 10$, $k = 3$.

The pyramid has 4 corners, 6 edges, 4 surfaces.

Each corner has $1 \times 4 = 4$ or k classes.
" edge " $9 \times 6 = 54$ " $6(n-1)$ "
" surface " $36 \times 4 = 144$ " $4\frac{(n-1)n}{2}$ " ; or $(n-1)$th Triangular No.

Total number of classes = 202.

Each corner has $1 \times 4 = 4$ or k Combinations
Each edge " $1{,}022 \times 6 = 6{,}132$ or $6(2^n-2)$ "
Each inner surface " $55{,}980 \times 4 = 223{,}920$ or $4[3^n-3(2^n-2)-3]$ "

Total comb. of surface shell = 230,056.

The four triangles are to be lettered, ABC, ABD, ACD, DCB, as in Figs. 2 and 3.

Diag. 2, Table VIII. Similarly the cells of the second inner shell contain two of the missing letter, and are lettered as in Diag. 3, Table VIII. The third shell, when there is one, will contain 3, the fourth 4, etc., of the missing letter. Each inner shell will have four less cells on a side than the next outer one. Hence until $n = 4$

	D			D	5	4	3	2	A	
	6	2520			7560	12600	12600	7560	2	
	5	5040	7560			12600	16800	12600	3	
ABD + 1C	4	6300	12600	12600			12600	12600	4	ACD + 1B
	3	5040	12600	16800	12600			7560	5	
	2	2520	7560	12600	12600	7560			C	
	A	1	2	3	4	5	6	7	B	
	1	720	2520	5040	6300	5040	2520	720	7	
	2	2520	7560	12600	12600	7560	2520	2520	6	
	3	5040	12600	16800	12600	5040	7560	5040	5	
ABC + 1D	4	6300	12600	12600	6300	12600	12600	6300	4	DCB + 1A
	5	5040	7560	5040	12600	16800	12600	5040	3	
	6	2520	2520	7560	12600	12600	7560	2520	2	
	7	720	2520	5040	6300	5040	2520	720	1	
	C	7	6	5	4	3	2	1	D	

TABLE VIII, DIAG. 2.
First Inner Shell of Pyramid.
No. of classes = 74. No. of comb. = 591,720. 7 Nos. to calculate.

	D		D		A	
ABD + 2C	3	25200			C	ACD + 2B
	A	2	3	4	B	
	2	18900	25200	18900	4	
ABC + 2D	3	25200	25200	25200	3	DCB + 2A
	4	18900	25200	18900	2	
	C	4	3	2	D	

TABLE VIII, DIAG. 3.
Second Inner Shell of Pyramid.
No. of classes = 10. No. of comb. = 226,800. 2 Nos. to calculate.

Summary of Pyramid.

4	Classes of	1	Variation	having	4	Combinations.
54	" "	2	Variations	"	6,132	"
144	" "	3	"	"	223,920	"
84	" "	4	"	"	818,520	"
286	" "	all	"	"	1,048,576	"
	= 11th Pyramidal Number.					$=4^{10}$.

there will be no inner shells. The number of inner shells will always be $n/4$ disregarding the remainder. The first inner shell will be numbered from 1 to $n-3$, the second from 2 to $n-2\cdot3$, the third from 3 to $n-3\cdot3$; in general, if t is the number of the shell it will be numbered from t to $n-3t$. These inner shells we shall call for short the first tetra, second tetra, etc. They are not really tetrahedral in shape, being composed mostly of right-angled, instead of equilateral triangles, but it will be convenient to call them as designated.

In Table VIII is worked out this case for $n=10$. Diag. 1, which represents the typical surface triangle ABC, is the same as in Table IV. Only the interior portion, enclosed in heavy lines, is exactly repeated on the other three surfaces. The edges, as stated, are repeated six, the corners four times. In reading the triangles it must be remembered that the numbers along the edges refer to the letters that stand at the acute angles. For the inner shells one or two of the missing letter are to be added, and the remainder of the n things are then of the letter that stands at the right angle.

The general formula for calculating the number of combinations corresponding to any interior cell is $n!/A!B!C!D!$. We need only consider in each case the typical triangle $ABC+tD$. Since for the first tetra, $D=1$, the general formula becomes $\frac{n!}{B!C!1![n-(B+C+1)]!}$. If we give various values to B and C, and put them in their proper places in the triangle, we obtain Table IX. It is apparent that this table has all the same regularities as Table V, so that we could here also obtain the interior terms from the edge terms of either columns, lines or diagonals, by determining proper coefficients. But we do not yet know the edge terms. These must themselves be derived somehow. If in imagination we follow any diagonal of the tetra out beyond the latter to where it pierces the surface of the pyramid, we shall find that it ends in a term that is suitable for our calculation. This feat of the imagination may not seem so easy, but the following plan may help.

Suppose Table V composed of a horizontal layer of cubes. Then Table IX, also composed of a horizontal layer of cubes, is to be set down on top of it, so that its first term lies upon the second term,

TABLE IX.

Formulas for calculating Typical Triangle ABC+1D of First Inner Pyramidal Shell.

A	1	2	3	4	5	6	B
1	$\frac{n!}{1!\,1!\,1!\,(n-3)!}$	$\frac{n!}{2!\,1!\,1!\,(n-4)!}$	$\frac{n!}{3!\,1!\,1!\,(n-5)!}$	$\frac{n!}{4!\,1!\,1!\,(n-6)!}$	$\frac{n!}{5!\,1!\,1!\,(n-7)!}$	$\frac{n!}{6!\,1!\,1!\,(n-8)!}$	
2	$\frac{n!}{1!\,2!\,1!\,(n-4)!}$	$\frac{n!}{2!\,2!\,1!\,(n-5)!}$	$\frac{n!}{3!\,2!\,1!\,(n-6)!}$	$\frac{n!}{4!\,2!\,1!\,(n-7)!}$	$\frac{n!}{5!\,2!\,1!\,(n-8)!}$		
3	$\frac{n!}{1!\,3!\,1!\,(n-5)!}$	$\frac{n!}{2!\,3!\,1!\,(n-6)!}$	$\frac{n!}{3!\,3!\,1!\,(n-7)!}$	$\frac{n!}{4!\,3!\,1!\,(n-8)!}$			
4	$\frac{n!}{1!\,4!\,1!\,(n-6)!}$	$\frac{n!}{2!\,4!\,1!\,(n-7)!}$	$\frac{n!}{3!\,4!\,1!\,(n-8)!}$				
5	$\frac{n!}{1!\,5!\,1!\,(n-7)!}$	$\frac{n!}{2!\,5!\,1!\,(n-8)!}$					
6	$\frac{n!}{1!\,6!\,1!\,(n-8)!}$						
C							

Triangle is numbered from 1 to n-3. Table ends when n=B+C+1.

General Formula $\frac{n!}{B!\,C!\,1![n-(B+C+1)]!}$

second column, of Table V. The latter now twice repeated, but with one line removed, is to be supposed set up on edge so as to enclose the right angle of IX with two vertical walls. Then it

is not difficult to see that the diagonal of the first term of IX pierces these back walls in the third term of the second line. The next diagonal of IX, of course, meets the fourth term, etc. By

A	1	2	3	4	5	6	7	8	B
1	2	3	4	5	6	7	8	9	Natural Nos.
2	3	6	10	15	21	28	36		Triangular Nos.
3	4	10	20	35	56	84			Pyramidal Nos.
4	5	15	35	70	126				Etc.
5	6	21	56	126					
6	7	28	84						
7	8	36							
8	9								
C									

TABLE X.

Coefficients for Calculating Typical Triangle of First Inner Pyramidal Shell.

Table is of Indefinite Extent.

A	1	2	3	4	5	6	7	B
1	2× 360	3× 840	4× 1260	5× 1260	6× 840	7× 360	8× 90	
2	3× 840	6× 1260	10× 1260	15× 840	21× 360	28× 90		
3	4× 1260	10× 1260	20× 840	35× 360	56× 90			
4	5× 1260	15× 840	35× 360	70× 90				
5	6× 840	21× 360	56× 90					
6	7× 360	28× 90						
7	8× 90							
C								

TABLE XI.

Method of Calculating Triangle ABC of Diag. 2, Table VII, from Second Line of Surface Triangle and Coefficients of Table X.

comparison of the two tables it is in fact seen that the factor $n!/[n-(B+C+1)]!$, which is constant along the diagonals of IX, is the same in the surface terms in which they end. It is only necessary then to determine the proper coefficients. This can be done as before by factoring the general expression, thus:

$$\frac{n!}{B!\,C!\,1![n-(B+C+1)]!} =$$

$$\frac{n!}{(B+C)!\,1![n-(B+C+1)]!} \times \frac{(B+C)!}{B!\,C!} = {}_nC_{B+C} \times {}_{B+C}C_B$$

Giving B and C their various values as before, we obtain Table X of the coefficients. It is seen at once that this table is exactly the same as Table VI for the surface triangle, except that the first line and column are omitted. The calculation of these inner terms hence becomes extremely simple, and may be reduced to the following rule.

The mth line of the first tetra is derived from the 2d line of the surface triangle by discarding $m+1$ terms, and multiplying the remaining terms by the $(m+1)$-hedroidal numbers in order, beginning with the second.

A similar investigation will lead to a similarly simple result for the second tetra, which may be reduced to the following rule:

The mth line of the second tetra is derived from the 3d line of the surface triangle, by discarding $m+3$ terms, and multiplying the remaining terms by the $(m+2)$-hedroidal numbers in order, beginning with the third.

Similar rules may be derived for the succeeding tetra, but if we call t the number of the tetra we may combine them all in one general rule as follows:

The mth *line of the* tth *tetra is derived from the* $(t+1)$th *line of the surface triangle by discarding* $2t+m-1$ *terms and multiplying the remaining terms by the* $(t+m)$*-hedroidal numbers in order, beginning with the* $(t+1)$th.

This rule is general not only for all the inner tetras, but by putting t in it equal to 0 it reduces to the rule previously given for the surface triangle, which thus may be considered as the 0-tetra. This one rule hence covers all cases up to the present.

If we construct a series of pyramids, like that of Table VIII, for the successive values of n from 0 up, but give each a thickness of one cell in the direction of the *fourth* dimension, and pile the successive pyramids so that their A vertices are adjacent to each other in the direction of this dimension, then we shall obtain the four-dimensional arithmetical pyramid. Each three-dimensional pyramid will be a slice of the four-dimensional one, perpendicular to its fourth-dimensional axis, just as each two-dimensional diagram of Fig. 1 is a slice of the three-dimensional pyramid. Each cubical cell will now acquire a thickness equal to its edge in the direction of the fourth dimension and so become a four-dimensional cube, or teseract as it is sometimes called. The whole system will of course contain all classes of combinations up to four variations.

II.

In Part I we have dealt with the combinations of any number of things, each capable of 1, 2, 3 or 4 variations, and found that all possibilities could be represented by tables, having respectively 0, 1, 2 and 3 dimensions, viz., by the point, line, triangle and triangular pyramid. In each case we required a table of $k-1$ dimensions. Hence if we allow more than four variations we must, by the same rule, step out into space of higher dimensions, making use in each case of a $(k-1)$-dimensional pyramid.

Let us first take the case of $k=5$. Call the variations A, B, C, D and E. By reasoning exactly analogous to that of the case $k=4$, it is clear that from every ABCD cell of the three-dimensional pyramid can be developed a series of new cells equal to the number of A in that cell, by exchanging successively the A's for E's. The only proper place to put these new cells is to build them out from the respective ABCD cells from which they were developed, in the direction of the *fourth* dimension. Because of the regularly diminishing number of the A's in the cells, in passing outward from the A vertex toward the BCD plane, the new solid developed will have the form of a four-dimensional pyramid, analogous to the three-dimensional pyramid previously described. We shall call it a *pentahedroid*, or a *penta* for short, though it is really right-angled instead of equilateral. This penta, as shown by the fifth line of the arithmetical triangle, is bounded by 5 corners, 10 edges, 10 triangular surfaces and five tetrahedra, all enclosing an interior four-dimensional space. These configurations will carry respectively the classes of 1, 2, 3, 4 and 5 variations. The typical triangles of the classes of 1, 2, 3 and 4 variations will be exactly the same as before, except for the different number of repetitions. The five bounding tetras will have interior shells exactly the same as those of diagrams 2 and 3 of Table VIII, and these being independent of one another will be repeated in entirety five times. The 20 surface triangles of the 5 tetras however coincide in pairs, reducing to 10; the 30 edges coincide in threes, reducing to 10; the 20 corners coincide in fours, becoming 5, as already stated. In other words 4 instead of 3 edges now radiate from every vertex, 3 instead of 2 planes from every edge, while every plane divides 2 adjacent tetras from each other.

These 10 surface triangles and the interior shells of the 5

bounding tetras constitute the surface or zero shell of the pentahedroid. The interior space can be considered as before to be made up of concentric pentahedroidal shells, each one cell in thickness in the direction of the fourth dimension. Each such shell will be exactly similar to the surface shell. It will have the same number and kind of boundaries, and can hence be represented in just the same way, viz., by 10 surface triangles, and the interior shells of the 5 bounding tetrahedra. The latter will be called: the first inner tetra shell of the first inner penta shell, second tetra of first penta, etc.

Each inner penta shell will have five less cells on a side than the next outer shell. There will therefore be $n/5$, neglecting remainder, such inner shells. Each will contain one more of each of the two missing letters. The typical triangles, which we shall call the surface triangles of the inner pentas, will be lettered and numbered as follows:

1st penta,	ABC + 1D + 1E	1 to $n-4$
2d "	ABC + 2D + 2E	2 to $n-2\cdot4$
3d "	ABC + 3D + 3E	3 to $n-3\cdot4$
pth "	ABC + pD + pE	p to $n-4p$

The tetras of the inner pentas will be lettered and numbered as follows:

NAME OF TETRA	LETTERING	NUMBERING	No. OF INNER TETRA SHELLS
1st tetra of 1st penta	ABC+2D+1E	2 to $n-3-4$	$\frac{n-5}{4}$
2d " " 1 "	ABC+3D+1E	3 to $n-2\cdot3-4$	
tth " " 1 "	ABC+(t+1)D+1E	t+1 to $n-3t-4$	
1st " " 2d "	ABC+3D+2E	3 to $n-3-2\cdot4$	$\frac{n-2\cdot5}{4}$
2d " " " "	ABC+4D+2E	4 to $n-2\cdot3-2\cdot4$	
tth " " " "	ABC+(t+2)D+2E	t+2 to $n-3t-2\cdot4$	
tth " " pth "	ABC+(t+p)D+pE	t+p to $n-3t-4p$	$\frac{n-5p}{4}$

Of course for the other triangles all combinations of the five letters will be taken. This case for $n = 10$ is worked out in Table XII. Similar tables can be made for other values of n.

TABLE XII.

Pentahedroidal Pyramid for $n = 10$, $k = 5$.

(Four Dimensions.)

Boundaries: 5 Corners, 10 Edges, 10 Surfaces, 5 Tetrahedra.

Diag. 1. Surface Penta Shell, Surface Triangles.

Typical Triangle ABC, same as Diag. 1, Table VIII, but

5 Corners	× 1 cell each	=	5 Cells.
10 Edges	× 9 cells "	=	90 "
10 Surfaces	× 36 cells "	=	360 "
Total Surface Cells of Surface Shell			455 "

Each Corner	has	1 × 5	=	5	Combinations.
" Edge	"	1,022 × 10	=	10,220	"
" Surface	"	55,980 × 10	=	559,800	"
Total of Surface Triangles			=	570,025	"

The Ten Triangles are lettered,

ABC	ACD	ADE	BCD	BDE	CDE
ABD	ACE		BCE		
ABE					

Diag. 2. First Inner Tetra Shell of Surface Penta Shell.

Typical Triangle ABC + 1D, same as in Diag. 2, Table VIII.

The Shell contains 5 such Pyramids, hence

$5 \times 74 = 370$ Cells, and $5 \times 591{,}720 = 2{,}958{,}600$ Combinations.

The Five Pyramids are to be lettered,

ABCD ABCE ABDE ACDE BCDE

Each Pyramid is composed of 4 Triangles, making 20 in all for the Shell. Those of the first Pyramid ABCD are lettered,

ABC + 1D, ABD + 1C, ACD + 1B, BCD + 1A.

Similarly the other 4 Pyramids are lettered.

Diag. 3. Second Inner Tetra Shell of Surface Penta.

Typical Triangle ABC + 2D, same as Diag. 3, Table VIII.

This Diag. five times repeated gives,

$5 \times 10 = 50$ Cells, and $5 \times 226{,}800 = 1{,}134{,}000$ Combinations.

Lettering same as for first Shell.

Total of Inner Tetra Shells	4,092,600 Comb.
Total of entire Surface Penta Shell, ..	4,662,625 "

TABLE XII (Continued).

A	1	2	3	4	5	6	B
1	5040	15120	25200	25200	15120	5040	
2	15120	87800	50400	37800	15120		
3	25200	50400	50400	2500			
4	25200	37800	25200				
5	15120	15120					
6	5040						
C							

TABLE XII, DIAG. 4.
First Inner Penta Shell.
Typical Surface Triangle
ABC+1D+1E

5 Corners × 1 Cell each = 5 Cells
10 Edges × 4 Cells " = 40 "
10 Surfaces × 6 Cells " = 60 "
(Shown within the heavy lines)
Total = 105 "

5 Corners with 5,040 Combinations each = 25,200 Comb.
10 Edges " 80,640 " " = 806,400 "
10 Surfaces " 264,600 " " = 2,646,000 "
Total, of Surface Triangles, First Inner Penta = 3,477,600 "

Lettering.

The Ten Triangles of the First Inner Penta Shell are to be lettered thus:

ABC+1D +1E ACD+1B +1E ADE+1B +1C
ABD+1C +1E ACE+1B +1D
ABE+1C +1D
BCD+1A +1E BDE+1A +1C CDE+1A +1B
BCE+1A +1D

		D			
		3			
D	3	A	2	3	B
		2	75600	75600	3
		3	75600	75600	2
		C	3	2	D

DIAG. 5.
First Inner Tetra Shell of First Inner Penta Shell. Typical Triangle ABC+2D+1E.

Complete Diag. according to Fig. 2 Text.

The 5 Tetra contain 4×5=20 Cells and 75600×4×5=1,512,000 Comb.

No further Tetra Shells to the First Inner Penta.

Triangles are lettered ABC+2D+1E, DCB+2A+1E;
ACD+2B+1E, ABD+2C+1E are empty.
Similarly the other Four Tetras.

TABLE XII (Concluded).

A	2	B
2	113400	
C		

DIAG. 6.

Second Inner Penta Shell. **Typical Triangle** ABC+2D+2E.

This Shell has but One Cell (since $n/5=2$) and contains 2A, 2B, 2C, 2D, 2E.

Hence Total Combinations of Second Penta Shell = $10!/(2!)^5=113,400$.
Total of Inner Pentas (Diags. 4, 5 and 6) =5,103,000 Comb.

Summary of Pentahedroid.

5	Classes of	1	Variation	having	5	Combinations.
90	"	" 2	Variations	"	10,220	"
360	"	" 3	"	"	559,800	"
420	"	" 4	"	"	4,092,600	"
126	"	" 5	"	"	5,103,000	"
1001	"	" all	"	"	9,765,625	"
	= 11th Pentahedroidal Number.				$=5^{10}$.	

The general formula for any interior cell of the penta is $n!/A!B!C!D!E!$. For the surface triangles of the first inner penta this reduces to $\frac{n!}{B!C!1!1![n-(B+C+2)]!}$. Assigning B and C their various values, we obtain Table XIII, for the typical triangle, lettered ABC+1D+1E. Now the typical triangle of the first inner tetra of the surface pyramids is lettered ABC+1D, and its general formula is $\frac{n!}{B!C!1![n-(B+C+1)]!}$. It differs from the present only in the one E lacking. If we compare Table IX with XIII it is seen that the second column of the former coincides with the first column of the latter in the number of A's present in each cell, as shown by the last factor of the denominator. Further it is readily seen that every term of the latter is just double the corresponding term of the former. Comparing further the second column of XIII with the third of IX we find that every term of the former is treble the corresponding term of the latter, and so on for the following columns. We may therefore reduce this case to the following rule:

The surface triangle of the first inner penta shell is derived from the first inner tetra of the surface shell by discarding one column and multiplying the remaining columns successively by the natural numbers in order beginning with the second.

Going through a similar process for the surface triangles of the second penta, we should find that these are derived from the second tetra of the surface shell by discarding 2 columns and multiplying the remaining columns by the triangular numbers in order beginning with the third.

TABLE XIII.

Formulas for Calculating First Inner Penta Shell.

Typical Surface Triangle ABC+1D+1E

General Formula $\frac{n!}{B!C!1!1![n-(B+C+2)]!}$

Table Ends when $n=B+C+2$.

A	1	2	3	4	5	B
1	$\frac{n!}{1!1!1!1!(n-4)!}$	$\frac{n!}{2!1!1!1!(n-5)!}$	$\frac{n!}{3!1!1!1!(n-6)!}$	$\frac{n!}{4!1!1!1!(n-7)!}$	$\frac{n!}{5!1!1!1!(n-8)!}$	
2	$\frac{n!}{1!2!1!1!(n-5)!}$	$\frac{n!}{2!2!1!1!(n-6)!}$	$\frac{n!}{3!2!1!1!(n-7)!}$	$\frac{n!}{4!2!1!1!(n-8)!}$		
3	$\frac{n!}{1!3!1!1!(n-6)!}$	$\frac{n!}{2!3!1!1!(n-7)!}$	$\frac{n!}{3!3!1!1!(n-8)!}$			
4	$\frac{n!}{1!4!1!1!(n-7)!}$	$\frac{n!}{2!4!1!1!(n-8)!}$				
5	$\frac{n!}{1!5!1!1!(n-8)!}$					
C						

Finally in general we should find:

The surface triangle of the pth penta is derived from the pth surface tetra by discarding the first p columns and multiplying

the remaining columns by the $(p+1)$-hedroidal numbers in order, beginning with the $(p+1)$th.

Examining similarly the inner tetras of the penta shells we find that the first tetra of the first penta is lettered ABC + 2D + 1E, while the second tetra of the surface is lettered ABC + 2D, differing again only by the one E lacking. Hence the former may be derived from the latter in a manner similar to the surface triangles of the pentas. Without going through all the details it may at once be stated that the following general rules may easily be derived:

The tth tetra of the *first* inner penta is derived from the $(t+1)$th surface tetra by discarding the first column and multiplying the remaining columns by the natural numbers in order beginning with the $(t+2)$d.

The tth tetra of the *second* inner penta is derived from the $(t+2)$d surface tetra by discarding two columns and multiplying the remaining columns by the triangular numbers in order, beginning with the $(t+3)$d.

Finally we may set up the following perfectly general rule for any tetra:

The tth tetra of the pth penta is derived from the $(t+p)$ surface tetra by discarding the first p columns and multiplying the remaining columns successively by the $(p+1)$-hedroidal numbers in order, beginning with the $(t+p+1)$th.

Substituting in the above $t=0$, we get the rule for the surface triangles of any inner penta, previously given, so that this rule is perfectly general for all inner pentas and their attached tetras.

Finally let $k=6$, viz., A, B, C, D, E and F. This is the case which is presented by a number of dice, each one of which may fall on any one of its six faces. We shall now require for proper representation of all classes a *five*-dimensional pyramid, or hexahedroid, or hexa as we shall call it for short. From the sixth line of the arithmetical triangle we find that such a figure is bounded by 6 corners, 15 edges, 20 surfaces, 15 tetrahedra, 6 pentahedra, and contains one interior five-dimensional space. These will carry the classes of 1, 2, 3, 4, 5 and 6 variations respectively. The classes of 1, 2 and 3 variations will be represented by the 20 surface triangles, each exactly the same as the previous cases except that regard must be had for the proper number of repetitions of the edges and corners. The classes of 4 variations will be represented by the proper number of surface tetra shells, exactly similar to

NAME OF BOUNDARY	LETTERING	NUMBERING	NO. OF SHELLS	TRIANGLES PER SHELL
Surface Triangles of 1st Hexa	ABC+1D+1E+1F	1 to $n-5$		20
tth Tetra " " "	ABC+tD+1D+1F	$t+1$ " $n-3t-5$	$t=\frac{n-6}{4}$	15×4=60
Surface of 1st Penta " " "	ABC+2D+2D+1F	2 " $n-4-5$		6×10=60
tth Tetra" " " " " "	ABC+(t+2)D+2E+1F	$t+2$ " $n-3t-4-5$	$t=\frac{n-5-6}{4}$	6×5×4=120
Surface " pth " " " "	ABC+(p+1)D+(p+1)E+1F	$p+1$ " $n-4p-5$	$p=\frac{n-6}{5}$	6×10=60
tth Tetra" " " " " "	ABC+($t+p$+1)D+(p+1)E+1F	$t+p+1$ " $n-3t-4p-5$	$t=\frac{n-5p-6}{4}$	6×5×4=120
Surface Triangles " h "	ABC+hD+hE+hF	h " $n-h$	$h=\frac{n}{6}$	20
tth Tetra " " "	ABC+($t+h$)D+hE+hF	$t+h$ " $n-3t-5h$	$t=\frac{n-6h}{4}$	15×4=60
Surface of pth Penta " " "	ABC+($p+h$)D+($p+h$)E+hF	$p+h$ " $n-4p-5h$	$p=\frac{n-6h}{5}$	6×10=60
tth Tetra" " " " " "	ABC+($t+p+h$)D+($p+h$)E+hF	$t+p+h$ " $n-3t-4p-5h$	$t=\frac{n-5p-6h}{4}$	6×5×4=120

TABLE XIV. Lettering and Numbering of a Hexahedroid.

Diagrams 2 and 3 of Table VIII but 15 times repeated for the 15 bounding tetras. The classes of 5 variations will be represented by the proper number of surface penta shells, with their accompanying inner tetra shells, exactly similar to Diagrams 4, 5 and 6 of Table XII, but each 6 times repeated for the 6 bounding pentahedroids. Hence it remains only to consider the cells of the interior five-dimensional space. As before, we shall consider the interior to be made up of concentric inner hexa shells, each one cell in thickness in the direction of the fifth dimension. Each of these inner shells will have the same number and kind of boundaries as the surface or zero hexa just described, and will therefore be represented by the same series of diagrams, viz., 20 surface triangles, with their 15 edges and 6 corners, 15 surface tetra, with their inner shells, 6 bounding pentas, each in turn represented as in Table XII, by 10 surface triangles, with their 10 edges and 5 corners, and by five bounding tetras with their inner shells. Each inner hexa will have six cells less on a side than the next outer one. The number of such inner hexa shells will therefore be $n/6$, neglecting remainder.

The lettering and numbering of the surface triangles, tetras and pentas, with the tetras of the latter, is the same as in the previous case. Hence it only remains to show the numbering and lettering of the inner hexa shells. This is done in Table XIV. The whole table can be developed from the general formulas of the last line by substituting the proper values of t, p and h. In fact by substituting 0 for any of them these formulas will give the surface configurations, and hence the pentahedroid of the previous case. For example, if we put them all equal to zero we get that the surface triangle is lettered ABC and numbered from 0 to n. Also it will have $n/3$ inner triangular shells.

It remains to consider how the combinations for each interior class may be calculated. Without going through the details we may at once state that a perfectly general rule may be set up as follows:

The tth tetra of the pth penta of the hth hexa is derived from the $(t+p)$ tetra of the hth penta by discarding the first $(p+h$ columns and multiplying the remaining columns successively by th $(p+h+1)$-hedroidal numbers in order, beginning with the $(t+p$ $h+1)$th.

	Values to Assign				Name of Shell	Lettering	Numbering	Derive From Shell	Discard	Multiply Remai'g by	Begin With	Number of Inner Shells
Diag. No.	t	p	h	r	tth Tetra of pth Penta of hHexa	ABC+$(t+p+h)$D +$(p+h)$E+hF	$t+p+h$ to $n-3t-4p-5h$	$(t+p)$ Tetra of h Penta	$p+h$ Cols.	$(1+p+h)$ hed'id Nos.	the $1+t+p+h$	$\frac{n-5p-6h}{r}$
1	0	0	0	6	Surface Triangles of 0 Hexa	ABC	0 to n $=10$	10th Line Arith. Triang				$\frac{n}{6}=1$ Hexa
2	1	0	0	4	1st Tetra of 0 Hexa	ABC+1D	1 " $n-1\cdot3=n-3=7$	2d Line Surface Triangle	2 Terms	Nat. Trian. Pyr. Etc.	2d	$\frac{10}{4}=2$ Tetra
3	2	0	0		2d Tetra of 0 Hexa	ABC+2D	2 " $n-2\cdot3=n-6=4$	3d Line Surface Triangle	4 Terms	Nat. Trian. Pyr. Etc.	3d	
4	0	1	0	5	1 Penta of 0 Hexa	ABC+1D+1E	1 " $n-1\cdot4=n-4=6$	1st Tetra	1 Col.	Natural	2d	$\frac{10}{5}=2$ Penta
5	1	1	0	4	1st Tetra of 1st Penta of 0 Hexa	ABC+2D+1E	2 " $n-1\cdot3-1\cdot4$ $=n-7$ $=3$	2d Tetra	1 Col.	Natural	3d	$\frac{10-1\cdot5}{4}=1$ Tetra.
6	0	2	0		2d Penta of 0 Hexa	ABC+2D+2E	2 " $n-2\cdot4=n-8=2$	2d Tetra	2 Cols.	Triangular	3d	
7	0	0	1		Surface Triangles of 1st Hexa	ABC+1D+1E+1F	1 " $n-1\cdot5=n-5=5$	1st Penta	1 Col.	Natural	2d	
8	1	0	1	4	1st Tetra of 1st Hexa	ABC+2D+1E+1F	2 " $n-1\cdot3-1\cdot5$ $=n-8$ $=2$	1st Tetra of 1st Penta	1 Col.	Natural	3d	$\frac{10-1\cdot6}{4}=1$ Tetra

Table XV. Preliminary Scheme, Hexahedroid for $n=10$, $k=6$.
Derived from General Formulas. (Five Dimensions).

Each Hexa is bounded by 6 Corners, 15 Edges, 20 Surfaces, 15 Tetrahedra, 6 Pentahedroids.
" Penta " " " 5 " 10 " 10 " 5 "
" Tetra " " " 4 " 6 " 4 "

TABLE XV (Concluded).

DIAG. 8.

A	2	B
2	226800	
C		

First Inner Tetra of First Inner Hexa. Typical Triangle ABC+2D+1E+1F.

One Cell only but 15 Tetra, giving Total of 15 Cells.
15×226,800=3,402,000 Total Combinations.

The 15 tetras are lettered same as in Diag. 2. In the single cell composing each tetra are contained two each of the four letters designating the tetra, and one of each of the two missing letters.

Total Combinations of First Inner Hexa Shell
= Sum of Diags. 7 and 8 = 16,435,440.

Summary of Hexahedroid.

6	Classes	of	1	Variation	having	6	Combinations
135	"	"	2	Variations	"	15,330	"
720	"	"	3	"	"	1,119,600	"
1260	"	"	4	"	"	12,277,800	"
756	"	"	5	"	"	30,618,000	"
126	"	"	6	"	"	16,435,440	"
3003	"	"	all	"	"	60,466,176	"
= 11th Hexahedroidal Number.						$= 6^{10}$.	

By putting h equal to zero in this rule it reduces to the one already given for the pentahedroid.

In Table XV is worked out from the general formulas a hexahedroid for $n = 10$. First is given a preliminary table showing the number and kind of diagrams needed. The first line of this table repeats the general formulas from which the whole is derived. No really new diagrams are required until we reach the first inner hexa, and only the surface triangles and the first tetra shell of this, the latter containing too only 1 cell, are developed. It might perhaps be more interesting to use a higher value of n so as to develop more of the inner shells, but the numbers increase so rapidly in size that space forbids. For example, if we used $n = 15$ the total of all the combinations would be $6^{15} = 470,184,984,576$ and 16 diagrams would be required.

Let $k = 7$ or higher. We might go on giving k successively higher values, and so develop a septa, an octa, a nona, etc. But the methods would always be the same, and in every case we should end with a general rule that included all of the previous ones. Hence we may at once give the perfectly general rule that will include all the preceding and all the succeeding, viz.:

The typical triangle of the t tetra, of the p penta, of the h

hexa, of the s septa, of the o octa,.....of the $q(k-1)$-hedroidal shell, of the $f(k)$-hedroidal shell.

(1) *will be lettered* $\mathrm{ABC}+(t+p+h+s+o+\ldots\ldots+f)\mathrm{D}+(p+h+s+o+\ldots\ldots+f)\mathrm{E}+(h+s+o+\ldots\ldots+f)\mathrm{F}+(s+o+\ldots\ldots+f)\mathrm{G}+\ldots\ldots+f$ times the kth letter.

(2) *will be numbered* from $t+p+h+s+o+\ldots\ldots+f$ to $n-3t-4p-5h-6s-7o-\ldots\ldots-(k-1)f$.

(3) *will be derived from* the typical triangle of the $(t+p)$ tetra of the h penta, of the s hexa, of the o septa,..... .....of the $q(k-1)$-hedroidal shell, by discarding the first $p+h+s+o+\ldots\ldots+f$ columns and multiplying the remaining columns successively, by the $(1+p+h+s+o+\ldots\ldots+f)$-hedroidal numbers in order, beginning with the $(1+t+p+h+s+o+\ldots\ldots+f)$th.

(4) *and will have on each edge,* $n-4t-5p-6h-7s-8o\ldots\ldots-kf+1$ cells.

(5) The number of r-hedroidal shells required will be $\dfrac{n-5p-6h-7s-8o\ldots\ldots-fk}{r}$, where for r is to be substituted the order of the shell required, and the corresponding letter of the shell in the numerator is then to be omitted.

To apply these rules simply give to the letters t, p, h, s, etc., successively the values, 0, 1, 2, 3, etc., in all combinations, until negative values occur, or until the proper number of shells have been developed. As far as lettering and numbering are concerned, these rules apply to all cases. For derivation they apply only to the inner shells after the surface tetra. The latter and the surface triangles must be calculated line by line, according to the general rule given on page 21. By considering the surface triangles and tetra to be made up of triangular shells, and considering a typical *edge* of such shells, calling the outer edge the zero shell, a perfectly general rule could be given for all cases. But it would be cumbrous, so that practically we find it better to divide the derivation, as has been done, into two rules.

One may well question whether all the foregoing is very important or useful. Certainly it is not of very great advantage until high values of n and k are reached. Still even in fairly simple cases it is of some help. To show this, Table XVI has been given for $n=4$, $k=6$. This shows all the ways in which four dice may be

thrown. Here we reach only the *first* surface tetra, and even this has only one cell, viz., the case where all four of the dice show a different number. The number of ways in which this can occur is given directly by $n! = 24$. All the other classes are shown in the surface triangles. There are 6 where one number only appear, 45 where two appear, 60 where three different numbers appear, and 15 where all four dice show different numbers. All the calculations can be made mentally, for when in the surface triangle we have said $6 \times 2 = 12$, we have obtained all of the different numbers. The method of representation enables all the classes to be enumerated without difficulty or doubt, and gives all the detailed information that can be desired. The total number of classes is 126, or the fifth hexahedroidal number. The total of all combinations is $1296 = 6^4$.

I	0	1	2	3	4	II
0	1	4	6	4	1	
1	4	12	12	4		
2	6	12	6			
3	4	4				
4	1					
III						

TABLE XVI.

Hexahedroid for $n=4$, $k=6$.

Diag. 1. Surface Triangles.
Typical Triangle I, II, III.

6 Corners	× 1 =	6 Comb.	6×1 =	6 Classes
15 Edges	×14 = 210	"	15×3 = 45	"
20 Surfaces	×36 = 720	"	20×3 = 60	"
Total	= 936	"	= 111	"

I	1	II
1	24	
III		

Diag. 2. First Surface Tetra Shell.
Typical Triangle I, II, III, + 1 IV.

15 Tetras × 24	= 360 Comb.	15×1 =	15 Classes
Total of All	= 1296 "	=	126 "
	= 6^4	= 5th Hexahedroidal No.	

In fact, it was the inquiry of a friend with regard to dice that started the whole investigation.

But whether one concedes to this system any measure of usefulness or not, it affords a striking example of the wonderful interrelations between numbers and geometry, and adds another to the many remarkable properties of the arithmetical triangle. Pascal, in the work referred to at the beginning, exclaimed: *"C'est une chose étrange combien il est fertile en propriétés! Chacun peut s'y exercer."* We have exercised ourselves there, and hope that this new property or extension may not prove wholly uninteresting.

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GENERAL RULE FOR CONSTRUCTING ORNATE MAGIC SQUARES OF ORDERS $\equiv 0$ (MOD. 4).

Take a square lattice of order $4m$ and draw heavy lines at every fourth vertical bar and also at every fourth horizontal bar, thus dividing the lattice into m^2 subsquares of order 4. The "period" consists of the $4m$ natural numbers 1, 2, 3.... $4m$. Choose from these *any* two pairs of complementary numbers, that is, pairs whose sum is $4m+1$ and arrange these four numbers, four times repeated, as in a Jaina square (first type) in the left-hand square of the top row of subsquares in the large lattice. It is essential that the Jaina pattern shall contain only one complementary couplet in each of

a_1	b_1	a_2	b_2
a_2	b_2	a_1	b_1
a_1	b_1	a_2	b_2
a_2	b_2	a_1	b_1

Fig. 1.

10	51	15	54	12	49	13	56
23	46	18	43	21	48	20	41
50	11	55	14	52	9	53	16
47	22	42	19	45	24	44	17
26	35	31	38	28	33	29	40
7	62	2	59	5	64	4	57
34	27	39	30	36	25	37	32
63	6	58	3	61	8	60	1

Fig. 2.

its four columns, i. e., if the two pairs are a_1 a_2 and b_1 b_2, every column must consist entirely of a's, or entirely of b's. The first Jaina type can be obtained by using the paths (1, 2), (2, 1) and the order a_1 b_1 a_2 b_2 four times repeated. This gives the square shown in Fig. 1, which fulfils the conditions. Proceed in the same way with each of the m subsquares in the top row, *using a different pair of complementaries in each subsquare.* Since the period 1, 2, 3.... $4m$ contains $2m$ complementary pairs and two pairs are used for each subsquare, it follows that when the top row of subsquares is filled up, all the $4m$ numbers will have been used.

Now fill all the remaining rows of subsquares in the large lattice with replicas of the top row. The *outline* so constructed can

always be turned over either of its central diagonals without repetition. The resulting square will therefore contain the first $(4m)^2$ numbers without repetition or omission, and it will always have the following magic properties.

A. *The Great Square*.....

1. is magic on its $4m$ rows and $4m$ columns;

2. is pandiagonal, i. e., magic on its $8m$ diagonals;

3. has Franklin's property of *bent diagonals* in an extended sense; i. e., we can start at any cell in the top row, and proceeding downward bend the diagonal at *any* heavy horizontal bar. It matters not how many times we bend, or at which of the heavy bars, providing only that when the traverse is completed, the number of cells passed over in the one direction (downward to the right) shall be exactly equal to the number passed over in the other direction (downward to the left). Similarly we may start at any cell in the left-hand column and, proceeding diagonally to the right, bend the diagonal at *any* heavy vertical bar under the same limitations.

2	3	7	6	4	1	5	8
7	6	2	3	5	8	4	1
2	3	7	6	4	1	5	8
7	6	2	3	5	8	4	1
2	3	7	6	4	1	5	8
7	6	2	3	5	8	4	1
2	3	7	6	4	1	5	8
7	6	2	3	5	8	4	1

Fig. 3.

8	48	8	48	8	48	8	48
16	40	16	40	16	40	16	40
48	8	48	8	48	8	48	8
40	16	40	16	40	16	40	16
24	32	24	32	24	32	24	32
0	56	0	56	0	56	0	56
32	24	32	24	32	24	32	24
56	0	56	0	56	0	56	0

Fig. 4.

It will be noticed that when the order of the square is $\equiv 4 \pmod 8$, i. e., when m is odd, the central bars are not *heavy bars*, and also the number of rows of subsquares is odd. We cannot therefore in these cases get a magic bent diagonal traverse from top to bottom of the square, but we may stop at the last heavy bar before reaching the bottom of the square, when we shall have a sum $4(m-1)$ times the mean, or we may carry the diagonal beyond the bottom of the square and traverse the top row of subsquares a

27	46	111	106	3	58	135	94	63	22	75	130
112	105	28	45	136	93	4	57	76	129	64	21
34	39	118	99	10	51	142	87	70	15	82	123
117	100	33	40	141	88	9	52	81	124	69	16
25	48	109	108	1	60	133	96	61	24	73	132
113	104	29	44	137	92	5	56	77	128	65	20
36	37	120	97	12	49	144	85	72	13	84	121
116	101	32	41	140	89	8	53	80	125	68	17
30	43	114	103	6	55	138	91	66	19	78	127
110	107	26	47	134	95	2	59	74	131	62	23
31	42	115	102	7	54	139	90	67	18	79	126
119	98	35	38	143	86	11	50	83	122	71	14

Fig. 5. S = 870

115	110	131	158	3	78	243	190	51	94	195	174	19	46	227	222
130	159	114	111	242	191	2	79	194	175	50	95	226	223	18	47
126	99	142	147	14	67	254	179	62	83	206	163	30	35	238	211
143	146	127	98	255	178	15	66	207	162	63	82	239	210	31	34
118	107	134	155	6	75	246	187	54	91	198	171	22	43	230	219
132	157	116	103	244	189	4	77	196	173	52	93	228	221	20	45
123	102	139	150	11	70	251	182	59	86	203	166	27	38	235	214
141	148	125	100	253	180	13	68	205	164	61	84	237	212	29	36
117	108	133	156	5	76	245	188	53	92	197	172	21	44	229	220
129	160	113	112	241	192	1	80	193	176	49	96	225	224	17	48
124	101	140	149	12	69	252	181	60	85	204	165	28	37	236	213
144	145	128	97	256	177	16	65	208	161	64	81	240	209	32	33
119	106	135	154	7	74	247	186	55	90	199	170	23	42	231	218
136	153	120	105	248	185	8	73	200	169	56	89	232	217	24	41
122	103	138	151	10	71	250	183	58	87	202	167	26	39	234	215
137	152	121	104	249	184	9	72	201	168	57	88	233	216	25	40

Fig. 6. S = 2056

second time, when the sum will be $4(m+1)$ times the mean. We can get in these cases a diagonal traverse $4m$ times the mean by inserting at any point one vertical series of four cells between any two heavy bars and then continuing diagonally.

4. The great square is 4-ply, and therefore 4-symmetrical, i. e., we may choose any vertical and any horizontal bar (not

1	382	20	399	3	384	18	397	5	386	16	395	7	388	14	393	9	390	12	391
40	379	21	362	38	377	23	364	36	375	25	366	34	373	27	368	32	371	29	370
381	2	400	19	383	4	398	17	385	6	396	15	387	8	394	13	389	10	392	11
380	39	361	22	378	37	363	24	376	35	365	26	374	33	367	28	372	31	369	30
41	342	60	359	43	344	58	357	45	346	56	355	47	348	54	353	49	350	52	351
80	339	61	322	78	337	63	324	76	335	65	326	74	333	67	328	72	331	69	330
341	42	360	59	343	44	358	57	345	46	356	55	347	48	354	53	340	50	352	51
340	79	321	62	338	77	323	64	336	75	326	65	334	73	327	68	332	71	329	70
81	302	100	319	83	304	98	317	85	306	96	315	87	308	94	313	89	400	92	311
120	299	101	282	118	297	103	284	116	295	105	286	114	293	107	288	112	291	109	290
301	82	320	99	303	84	318	97	305	86	316	95	307	88	314	93	309	90	312	91
300	119	281	102	298	117	283	104	296	115	285	106	294	113	287	108	292	111	289	110
121	262	140	279	123	264	138	277	125	268	136	275	127	268	134	273	129	270	132	271
160	259	141	242	158	257	143	244	156	253	145	246	154	253	147	241	152	251	149	250
261	122	280	139	263	124	278	137	265	126	276	135	267	128	274	133	269	130	272	131
260	159	241	142	258	157	243	144	256	155	245	146	254	153	247	148	252	151	249	150
161	222	180	239	163	224	178	237	165	226	176	235	167	228	174	233	169	230	172	231
200	219	181	202	198	217	183	204	196	215	185	206	194	213	187	208	192	211	189	210
221	162	240	179	228	164	238	177	226	166	236	175	227	168	234	173	229	170	232	171
220	199	201	182	218	197	203	184	216	195	205	186	214	193	207	188	212	191	209	190

Fig. 7.

necessarily heavy bars) and we shall find that any four cells, symmetrically situated with regard to these two bars as axes, will contain numbers whose sum is four times the mean. It follows that any $4m$ cells which form a symmetrical figure with regard to any such axes will contain numbers whose sum is the magic sum of the great square.

1	552	553	48	49	504	505	96	97	456	457	144	145	408	409	192	193	360	361	240	241	312	313	288
575	26	23	530	527	74	71	482	479	122	119	434	431	170	167	386	383	218	215	338	335	266	263	290
24	529	576	25	72	481	528	73	120	433	480	121	168	385	432	169	216	337	384	217	264	289	336	265
554	47	2	551	506	95	50	503	458	143	98	455	410	191	146	407	362	239	194	359	314	287	242	311
3	550	555	46	51	502	507	94	99	454	459	142	147	406	411	190	195	358	363	238	243	310	315	286
573	28	21	532	525	76	69	484	477	124	117	436	429	172	165	388	381	220	213	340	333	268	261	292
22	531	574	27	70	483	526	75	118	435	478	123	166	387	430	171	214	339	382	219	262	291	334	267
556	45	4	549	508	93	52	501	460	141	100	453	412	189	148	405	364	237	196	357	316	285	244	309
5	548	557	44	53	500	509	92	101	452	461	140	149	404	413	188	197	356	365	236	245	308	317	284
571	30	19	534	523	78	67	486	475	126	115	438	427	174	163	390	379	222	211	342	331	270	259	294
20	533	572	29	68	485	524	77	116	437	476	125	164	383	428	173	212	341	380	221	260	293	332	269
558	43	6	547	510	91	54	499	462	139	102	451	414	187	150	403	366	235	198	355	318	283	246	307
7	546	559	42	55	498	511	90	103	450	463	138	151	402	415	186	199	354	367	234	247	306	319	282
569	32	17	536	521	80	65	488	473	128	113	440	425	176	161	392	377	224	209	344	329	272	257	296
18	535	570	31	66	487	522	79	114	439	474	127	162	391	426	175	210	343	378	223	258	295	330	271
560	41	8	545	512	89	56	497	464	137	104	449	416	185	152	401	368	233	200	353	320	281	248	305
9	544	561	40	57	496	513	88	105	448	465	136	153	400	417	184	201	352	369	232	249	304	321	280
567	34	15	538	519	82	63	490	471	130	111	442	423	178	159	394	375	226	207	346	327	274	255	298
16	537	568	33	64	489	520	81	112	441	472	129	160	393	424	177	208	345	376	225	256	297	328	273
562	39	10	543	514	87	58	495	466	135	106	447	418	183	154	399	370	231	202	351	322	279	250	303
11	542	563	38	59	494	515	86	107	446	467	134	155	398	419	182	203	350	371	230	251	302	323	278
565	36	13	540	517	84	61	492	469	132	109	444	421	180	157	396	373	228	205	348	325	276	253	300
14	539	566	35	62	491	518	83	110	443	470	131	158	335	422	179	206	347	374	227	254	299	326	275
564	37	13	541	516	85	60	493	468	133	108	445	420	181	156	397	372	229	204	349	324	277	252	301

Fig. 8.

B. *The Subsquares.....*

5. are balanced Jaina squares, i. e., each of them has the 36 summations of a Jaina and in each case the magic sum is four times the mean number of the great square.

6. They have the property of subsidiary minors, i. e., if we erase any p rows of subsquares, and any p columns of the same and draw the remaining rows and columns together, we have a square with *all* the properties of the original great square.

EXAMPLES

In every case the Jaina pattern quoted above is used. Fig. 2 is an example of order 8 and the complementaries have been paired thus: 2,7 with 3,6; and 4,5 with 1,8. The La Hirean primaries of Fig. 2 are shown in Figs. 3 and 4.

* * *

Fig. 5 is an example of an order 12 square in which the pairing

1.16 — 2.15				13.4 — 14.3				12.5 — 11.6				8.9 — 7.10			
1	2	16	15	13	14	4	3	12	11	5	6	8	7	9	10
16	15	1	2	4	3	13	14	5	6	12	11	9	10	8	7
1	2	16	15	13	14	4	3	12	11	5	6	8	7	9	10
16	15	1	2	4	3	13	14	5	6	12	11	9	10	8	7

Fig. 9.

of the complementaries is 3,10 with 4,9; 1,12 with 5,8; and 6,7 with 2,11.

* * *

A square of order 16 is shown in Fig. 6. The couplets in this square are taken thus:

8 and 9 with 7 and 10; 1 and 16 with 5 and 12;
4 and 13 with 6 and 11; 2 and 15 with 3 and 14.

Figs. 7 and 8 show respectively squares of orders 20 and 24 in which the couplets are taken in numerical order, i. e., for order 20, 1 and 20 with 2 and 19; 3 and 18 with 4 and 17, etc.

In Fig. 8 there are 1008 magic diagonal summations. Since we can bend at any heavy bar, the number of bent diagonals from top to bottom, starting at a given cell in the top row, is the same as the

number of combinations of 6 things 3 at a time, viz., 20. Therefore there are $20 \times 24 = 480$ bent diagonals from top to bottom and 480 more from side to side. Adding the 48 continuous diagonals we get 1008.

In the foregoing pages the question of magic knight paths has not been considered. It is, however, easy for all orders > 8 and $\equiv 0$ (mod. 8) to add the knight nasik property *without sacrificing any of*

1	32	241	240	193	224	49	48	177	176	65	96	113	112	129	160
242	239	2	31	50	47	194	223	66	95	178	175	130	159	114	111
16	17	256	225	208	209	64	33	192	161	80	81	128	97	144	145
255	226	15	18	63	34	207	210	79	82	191	162	143	146	127	98
13	20	253	228	205	212	61	36	189	164	77	84	125	100	141	148
254	227	14	19	62	35	206	211	78	83	190	163	142	147	126	99
4	29	244	237	196	221	52	45	180	173	68	93	116	109	132	157
243	238	3	30	51	46	195	222	67	94	179	174	131	158	115	110
12	21	252	229	204	213	60	37	188	165	76	85	124	101	140	149
251	230	11	22	59	38	203	214	75	86	187	166	139	150	123	102
5	28	245	236	197	220	53	44	181	172	69	92	117	108	133	156
246	235	6	27	54	43	198	219	70	91	182	171	134	155	118	107
8	25	248	233	200	217	56	41	184	169	72	89	120	105	136	153
247	234	7	26	55	42	199	218	71	90	183	170	135	154	119	106
9	24	249	232	201	216	57	40	185	168	73	88	121	104	137	152
250	231	10	23	58	39	202	215	74	87	186	167	138	151	122	103

Fig. 10.

the other features, by a proper choice of the complementary couplets for the subsquare outlines. The example shown in Fig. 9 will explain. It shows the top row of subsquares in a scheme for order 16. The numbers above the squares indicate the couplets used, the Jaina pattern, Fig. 1, being used throughout. The rule is simple: the leading numbers, 1, 13, 12, 8 must sum four times the mean of the period, i. e., 34, while of course no one of them may be a complement of any other. Their complementaries 16,

4, 5, 9, will then have the same sum, and the second members in each square will be similarly related. The square is completed by filling the remaining rows with replicas and turning over a central diagonal. Fig. 10 is a square of order 16 constructed from the outline Fig. 9. It has all the properties of the 16^2 shown in Fig. 6, and is also magic on its 64 knight paths.

The following is an arrangement of the couplets for a square of order 24:

1.24–4.21	8.17–5.20	10.15–13.12	11.14–16.9	22.3–18.7	23.2–19.6

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ORNATE MAGIC SQUARES OF COMPOSITE ODD ORDERS.

When we consider these orders in the light of the general rule used for orders $\equiv 0 \pmod 4$ it appears at first sight that they cannot be made to fulfil all the conditions; but it is not essential to the *ply* property, nor to the balanced magic subsquares that the numbers be taken in complementary pairs for the subsquares of the outline. All that is necessary is that the groups of numbers chosen shall all have the same sum.

Suppose, as an illustration, we are dealing with order 15. If we can arrange the first 15 natural numbers in five balanced

2	7	15
7	15	2
15	2	7

Fig. 1.

2	6	12	11	9
15	13	8	3	1
7	5	4	10	14

Fig. 2.

columns, three in a column, and form five magic outlines of order 3, using a different column thrice repeated for each outline, we shall have five balanced magic outlines like Fig. 1. These can be arranged in the first row of subsquares with replicas in the following rows. If we can turn this outline upon itself in some way to avoid repetitions, we shall have a magic square which will be 9-ply and with magic subsquares. But will it be pandiagonal?

In the small outlines of 9 cells made from Fig. 1 as a pattern, it

will be noticed that like numbers must always occur in parallel diagonals; therefore if we arrange the five small squares so that like numbers always lie along / diagonals, the great outline will be "boxed" and therefore magic in \ diagonals, but in the / diagonals we shall have in every case only five different numbers each occurring thrice. The problem is thus reduced to finding a magic rectangle 3×5. We therefore construct such a rectangle by the method of "Complementary Differences"[1] as shown in Fig. 2.

In Fig. 3 we have the five magic outlines constructed from the five columns of the rectangle, and placed side by side with like

2	7	15	6	5	13	12	4	8	11	10	3	9	14	1
7	15	2	5	13	6	4	8	12	10	3	11	14	1	9
15	2	7	13	6	5	8	12	4	3	11	10	1	9	14

Fig. 3.

2	12	9	6	11	15	8	1	13	3	7	4	14	5	10
9	6	11	2	12	1	13	3	15	8	14	5	10	7	4
11	2	12	9	6	3	15	8	1	13	10	7	4	14	5
12	9	6	11	2	8	1	13	3	15	4	14	5	10	7
6	11	2	12	9	13	3	15	8	1	5	10	7	4	14

Fig. 4.

numbers always in the / diagonals, and so disposed that *the number* in any / diagonal is always succeeded (when the diagonal passes across into a neighboring square) by the number which succeeds it in its row in the rectangle.

If an associated square is required the magic rectangle must be associated and the large rectangle of subsquares must also be associated as a whole. It will be noticed that all these conditions will be fulfilled in practice if we write the successive columns of the magic rectangle Fig. 2 along the \ central diagonals of the successive square outlines in the larger rectangle Fig. 3, and fill in all the / diagonals with replicas. If now all the remaining rows of

[1] See "The Construction of Magic Squares and Rectangles by the Method of Complementary Differences," by W. S. Andrews, *Monist*, July, 1910, Vol. XX, No. 3.

subsquares be filled with replicas of the top row it will be found that the whole outline *cannot* be turned over either of its central diagonals without repetitions in the magic, but it *can* be turned successfully *in its own plane,* about its central point through one right angle, without repetitions. (This will bring the top row in coincidence with the left-hand column, so that the right-hand square in Fig. 3 is turned on its side and lies over the left-hand square.) The resulting magic is shown in Fig. 6. It is magic on its 15 rows,

155	28	171	125	88	156	20	178	126	80	163	21	170	133	81
44	211	114	14	181	39	224	106	9	194	31	219	119	1	189
139	98	57	199	68	147	94	53	207	64	143	102	49	203	72
157	30	167	127	90	152	22	180	122	82	165	17	172	135	77
40	213	116	10	183	41	220	108	11	190	33	221	115	3	191
140	103	51	200	73	141	95	58	201	65	148	96	50	208	66
164	16	174	134	76	159	29	166	129	89	151	24	179	121	84
34	218	117	4	188	42	214	113	12	184	38	222	109	8	192
142	105	47	202	75	137	97	60	197	67	150	92	52	210	62
160	18	176	130	78	161	25	168	131	85	153	26	175	123	86
35	223	111	5	193	36	215	118	6	185	43	216	110	13	186
149	91	54	209	61	144	104	46	204	74	136	99	59	196	69
154	23	177	124	83	162	19	173	132	79	158	27	169	128	87
37	225	107	7	195	32	217	120	2	187	45	212	112	15	182
145	93	56	205	63	146	100	48	206	70	138	101	55	198	71

Fig. 5. S = 1695

15 columns, 30 diagonals and 60 knight paths, also 9-ply and associated. The 25 subsquares of order 3 all sum 339 on their 3 rows and 3 columns. (It is easy to see that only one of them can have magic central diagonals, for a magic of order 3 can only have this property when it is associated, and in this case the mean number must occupy the central cell, but there is here only one mean number, viz., 113, therefore only the central subsquare can have magic diagonals.)

In exactly the same manner as above described, by using the

long rows of the magic rectangle Fig. 2, instead of the short columns, we can construct another ornate magic of order 15.

Fig. 4 shows the first row of 25-celled subsquares constructed from the *rows* of the rectangle, and using a magic square of order 5 as pattern. If we fill the two remaining rows of subsquares with replicas the outline can be turned over either of its central diagonals. The resulting square is shown in Fig. 7. It is magic on 15 rows, 15 columns, 30 diagonals and 60 knight paths, also 25-ply and asso-

2	127	210	6	125	208	12	124	203	11	130	198	9	134	196
202	15	122	200	13	126	199	8	132	205	3	131	209	1	129
135	197	7	133	201	5	128	207	4	123	206	10	121	204	14
32	157	150	36	155	148	42	154	143	41	160	138	39	164	136
142	45	152	140	43	156	139	38	162	145	33	161	149	31	159
165	137	37	163	141	35	158	147	34	153	146	40	151	144	44
107	172	60	111	170	58	117	169	53	116	175	48	114	179	46
52	120	167	50	118	171	49	113	177	55	108	176	59	106	174
180	47	112	178	51	110	173	57	109	168	56	115	166	54	119
182	82	75	186	80	73	192	79	68	191	85	63	189	89	61
67	195	77	65	193	81	64	188	87	70	183	86	74	181	84
90	62	187	88	66	185	83	72	184	78	71	190	76	69	194
212	22	105	216	20	103	222	19	98	221	25	93	219	29	91
97	225	17	95	223	21	94	218	27	100	213	26	104	211	24
30	92	217	28	96	215	23	102	214	18	101	220	16	99	224

Fig. 6 S = 1695

ciated. Also the nine subsquares of order 5 are balanced nasiks, summing 565 on their 5 rows, 5 columns and 10 diagonals.

The above method can of course be used when the order is the square of an odd number, e. g., orders 9, 25, etc. These have previously been dealt with by a simpler method which is not applicable when the order is the product of different odd numbers.

A similar distinction arises in the case of orders $\equiv 0 \pmod 4$ previously considered. These were first constructed by a rule which applied only to orders of form 2^m, e. g., 4, 8, 16, 32, etc., but the general rule is effective in every case.

There are two other ornate squares of order 15, shown in Figs. 5 and 8, these four forms of ornate squares being numbered in ascending order of difficulty in construction. Fig. 5 is constructed by using the paths $\begin{Bmatrix} 3, 5 \\ 5, 3 \end{Bmatrix}$ and taking the period from the *continuous diagonal* of the magic rectangle Fig. 2.

Fig. 5 is magic on 15 rows, 15 columns, 30 diagonals, 60 knight paths, and is 9-ply, 25-ply and associated.

The square shown in Fig. 8 has been only recently obtained;

17	132	153	171	86	30	128	151	178	78	22	124	164	170	85
174	81	26	122	162	166	88	18	135	158	179	80	25	127	154
131	152	177	84	21	123	165	173	76	28	130	157	169	89	20
87	24	126	161	167	83	16	133	153	180	79	29	125	160	172
156	176	77	27	129	163	168	90	23	121	155	175	82	19	134
212	12	39	111	191	225	8	31	118	183	217	4	44	110	190
114	186	221	2	42	106	193	213	15	38	119	185	220	7	34
11	32	117	189	216	3	45	113	181	223	10	37	109	194	215
192	219	6	41	107	188	211	13	33	120	184	224	5	40	112
36	116	182	222	9	43	108	195	218	1	35	115	187	214	14
92	207	144	51	71	105	203	136	58	63	97	199	149	50	70
54	66	101	197	147	46	73	93	210	143	59	65	100	202	139
206	137	57	69	96	198	150	53	61	103	205	142	49	74	95
72	99	201	145	47	68	91	208	138	60	64	104	200	145	52
141	56	62	102	204	148	48	75	98	196	140	55	67	94	209

Fig. 7. S = 1695

for many years the conditions therein fulfilled were believed to be impossible. It is magic on 15 rows, 15 columns and 30 diagonals, and is 3×5 rectangular ply, i. e., *any* rectangle 3×5 with long axis horizontal contains numbers whose sum is the magic sum of the square. Also the 15 subrectangles are balanced magics, summing 565 in their three long rows and 339 in their five short columns. This square is not associated, and only half of its knight paths are magic.

The three squares of order 15, shown in Figs. 5, 6 and 7, are

described as magic on their 60 knight paths, but actually they are higher nasiks of Class II, as defined at the end of my pamphlet on *The Theory of Path Nasiks.*[2] Further, the squares in Figs. 6 and 7 have the following additional properties.

Referring to the square in Fig. 7 showing subsquares of order 5; if we superpose the diagonals of these subsquares in the manner described in my paper on "Fourfold Magics" (*The Monist*, Vol. XX, p. 618, last paragraph), we obtain three magic parallelopipeds

37	93	191	81	163	32	99	185	89	160	45	102	188	78	151
167	219	5	59	115	180	222	8	49	106	172	213	11	51	118
135	27	143	199	61	127	18	146	201	73	122	24	140	209	70
97	183	86	136	43	92	189	80	164	40	105	192	83	154	31
212	9	50	119	175	225	12	53	109	166	217	3	56	111	178
30	147	203	64	121	22	138	206	66	133	17	144	200	74	130
187	78	161	36	103	182	84	155	44	100	195	87	158	34	91
2	54	110	179	220	15	57	113	169	211	7	48	116	171	223
150	207	68	124	16	142	198	71	126	28	137	204	65	134	25
82	153	41	96	193	77	159	35	104	190	90	162	38	94	181
47	114	170	224	10	60	117	173	214	1	52	108	176	216	13
210	72	128	19	136	202	63	131	21	148	197	69	125	29	145
157	33	101	180	88	152	39	95	194	85	165	42	98	184	76
107	174	215	14	55	120	177	218	4	46	112	168	221	6	58
75	132	23	139	195	67	123	26	141	208	62	129	20	149	205

Fig. 8. S = 1695

$5 \times 5 \times 3$. Denoting each subsquare by the number in its central cell, the three parallelopipeds will be:

I. 53, 169, 117.
II. 177, 113, 49.
III. 109, 57, 173.

These three together form an octahedroid $5 \times 5 \times 3 \times 3$ which is associated and magic in each of the four directions parallel to its edges.

If we deal in like manner with Fig. 6 which has subsquares of

order 3 we obtain five magic parallelopipeds of order $3\times3\times5$ together forming an associated magic octahedroid of order $3\times3\times5\times5$. Since the lengths of the edges are the same as those of the octahedroid formed from Fig. 7 square, these two four-dimensional figures are identical but the distribution of the numbers in their cells is not the same. They can however be made completely identical both in form and distribution of numbers by a slight change in our method of dealing with the square Fig. 6, i. e., by taking the square plates to form the parallelopipeds from the knight paths instead of the diagonals. Using the path $-1, 2$ we get 225, 106, 3, 188, 43 for the first plates of each parallelopiped, and then using $2, -1$ for the successive plates of each, we obtain the parallelopipeds:

I.	225,	8,	31,	118,	183
II.	106,	193,	213,	15,	38
III.	3,	45,	113,	181,	223
IV.	188,	211,	13,	33,	120
V.	43,	108,	195,	218,	1

This octahedroid is completely identical with that previously obtained from Fig. 7, as can be easily verified by taking any number at random and writing down the four series of numbers through its containing cell parallel to the edges, first in one octahedroid and then in the other. The sets so obtained will be found identical.

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PANDIAGONAL-CONCENTRIC MAGIC SQUARES OF ORDERS $4m$.

These squares are composed of a central pandiagonal square surrounded by one or more bands of numbers, each band, together with its enclosed numbers, forming a pandiagonal magic square.

The squares described here are of orders $4m$ and the bands or borders are composed of double strings of numbers. The central square and bands are constructed simultaneously instead of by the usual method of first forming the nucleus square and arranging the bands successively around it.

[2] *The Theory of Path Nasiks*, by C. Planck, M.A., M.R.C.S., printed by A. J. Lawrence, Rugby, Eng.

A square of the 8th order is shown in Fig. 1, both the central 4^2 and 8^2 being pandiagonal. It is 4^2 ply, i. e., any square group of 16 numbers gives a constant total of $8(n^2+1)$, where n = the number of cells on the edge of the magic. It is also magic in all of its Franklin diagonals; i. e., each diagonal string of numbers bending at right angles on either of the horizontal or vertical center lines of the square, as is shown by dotted lines, gives constant totals. In any size concentric square of the type here described, all of its concentric squares of orders $8m$ will be found to possess the Franklin bent diagonals.

The analysis of these pandiagonal-concentric squares is best illustrated by their La Hirean method of construction, which is

45	28	35	22	47	26	33	24
49	8	63	10	51	6	61	12
31	42	17	40	29	44	19	38
3	54	13	60	1	56	15	58
46	27	36	21	48	25	34	23
50	7	64	9	32	5	62	11
32	41	18	39	30	43	20	37
4	53	14	59	2	55	16	57

Fig. 1.

here explained in connection with the 12th order square. The square lattice of the subsidiary square, Fig. 2, is, for convenience of construction, divided into square sections of 16 cells each. In each of the corner sections (regardless of the size of the square to be formed) are placed four 1's, their position to be as shown in Fig. 2. Each of these 1's is the initial number of the series 1, 2, 3,.....$(n/4)^2$, which must be written in the lattice in natural order, each number falling in the same respective cell of a 16-cell section as the initial number. Two of these series are indicated in Fig. 2 by circles enclosing the numbers, and inspection will show that each of the remaining series of numbers is written in the lattice in the same manner, though they are in a reversed or reflected order. Any size subsidiary square thus filled possesses all the magic features of the final square.

In fact any group or string of numbers in these squares, that is symmetrical to the horizontal or vertical center line of the magic and is selected in accordance with the magic properties of the 16-cell subsidiary square, will give the sum $[r(n^2+1)]/2$, where r = the number of cells in the group or string, and n = the number of cells in the edge of the magic. One of these strings is exemplified in Fig. 5 by the numbers enclosed in circles.

1	224	61	228	5	220	57	232	9	216	53	236	13	212	49	240
113	176	77	148	117	172	73	152	121	168	69	156	125	164	65	160
205	20	241	48	201	24	245	44	197	28	249	40	193	32	253	36
189	100	129	96	185	104	133	92	181	108	137	88	177	112	(141)	84
2	223	62	227	6	219	58	231	10	215	54	(235)	(14)	211	50	(239)
114	175	78	147	118	171	74	151	122	167	(70)	155	126	(163)	(66)	(159)
206	19	242	47	202	23	246	43	198	(27)	250	39	194	31	254	35
190	99	130	95	186	103	134	91	182	(107)	138	87	178	111	142	83
3	222	63	226	7	218	59	230	11	(214)	55	234	15	210	51	238
115	174	79	146	119	170	75	150	123	(166)	71	154	127	162	67	158
207	18	243	46	203	22	247	42	199	26	(251)	38	195	(30)	(255)	(34)
191	98	131	94	187	102	135	90	183	106	139	(86)	(179)	110	143	(82)
4	221	64	225	8	217	60	229	12	213	56	233	16	209	(52)	237
116	173	80	145	120	169	76	149	124	165	72	153	128	161	68	157
208	17	244	45	204	21	248	41	200	25	252	37	196	29	256	33
192	97	132	93	188	101	136	89	184	105	140	85	180	109	144	81

Fig. 5.

To explain what is meant above in reference to selecting the numbers in accordance with the magic properties of the 16-cell subsidiary square, note that the numbers, 27, 107, 214, 166, in the exemplified string, form a magic row in the small subsidiary square, 70, 235, 179, 30 and 251, 86, 14, 163 form magic diagonals, and 66, 159, 255, 34 and 141, 239, 82, 52 form ply groups.

HARRY A. SAYLES.

SCHENECTADY, N. Y.

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THE MONIST

GOTTFRIED WILHELM LEIBNIZ.

(1646-1716.)

THIS number of *The Monist* is devoted to a commemoration of the scientific and philosophical work of Leibniz and its influences on modern thought. It is just two hundred years since Leibniz died, and thus it is fitting, as well as useful, that we should all remember just now rather particularly the mortal Leibniz and his undying work. The articles here outlined for this and the following issue of *The Monist* have been collected and prepared under the editorship of Mr. P. E. B. Jourdain, an English scholar well known to *Monist* readers through his many recent contributions on the subjects of physics and logic.

The first article is an account of Leibniz's life and work by C. Delisle Burns, and it gives a view of the various activities of Leibniz which are of general interest, and particularly the great part he took in the founding of academies. A description of Leibniz's logic by Philip E. B. Jourdain then follows. It has become more and more recognized of late years that logic was at the foundation of both Leibniz's mathematics and his metaphysics, and we have a most instructive example of the intimate connection of his logical and mathematical ideas when we study Leibniz's early mathematical manuscripts, which were published long after his death and are here translated by J. M. Child for the first time. Another article of

importance in connection with Leibniz's mathematics is Prof. Florian Cajori's account of his binary system of numeration that he held in great affection as leading to an arithmetic which was an "image of creation."

The influence of Descartes on Leibniz's philosophy is studied by C. Delisle Burns, and the influences that formed Leibniz's monadism are dealt with by T. Stearns Eliot. The author last mentioned also writes on the analogy between Leibniz's monads and the "finite centers" of Bradley's monism.

The last article brings us to Leibniz's modern influences. The logical influence of Leibniz on Lambert and later writers is touched upon in the above article on Leibniz's logic. It is seen also in a study of Bolzano by Miss Dorothy Maud Wrinch which will follow in the same connection in the January number of *The Monist*. It is conspicuous in the modern work of Frege, and of Peano, Russell and Couturat. It must be remembered of course, that the splendid work of Frege, which was almost wholly unaffected by any other logician but Leibniz, has combined with the work of Peano to influence the modern school of mathematical logicians.

A realization was given to part of Leibniz's ideal by Hermann Grassmann. It was intended that this number of *The Monist* should also celebrate the seventieth anniversary of Grassmann's prize for an essay on the connection of his geometrical analysis with Leibniz's Characteristic, which was awarded in 1846 by the Jablonowski Society of Leipsic. But this must be deferred until January. Then we shall present three articles by A. E. Heath. The first will be a critical sketch of the life and work of one whose writings contain the germ of many modern developments in mathematics and mathematical physics. Grassmann shared with Thomas Young the distinction of winning fame in both philology and mathematics. His biog-

raphy shows him as a homely and lovable man of wide interests, possessing to the last indomitable energy and unshaken faith in the power of his work. In the second article an analysis will be made of the factors which were and are at the root of the neglect of the work not only of Grassmann but also of all writers on the abstract questions of a basic science of form. The third article will show how Grassmann, starting from the geometrical Characteristic of Leibniz, applied the principles of his work previously published in 1844 to the realization of a true geometrical analysis. The author claims that in this analysis we have a complete fulfilment of the high hopes of Leibniz, and shows the relation of their work to modern non-metrical geometries and to symbolic analysis.

Portraits of Leibniz, Lambert, Bolzano, Grassmann, Frege, Peano and Russell, and some details about these portraits, will appear in the current (October) number of *The Open Court*.

The following gives the books most frequently cited in this number together with the abbreviations used throughout.

BIBLIOGRAPHY.

ABBREVIATIONS

Cantor: Moritz Cantor, *Vorlesungen über die Geschichte der Mathematik*. Vol. II, 2d ed., Leipsic, 1900; Vol. III, 2d. ed., Leipsic, 1901.

Couturat, 1901: Louis Couturat, *La Logique de Leibniz d'après des documents inédits*. Paris, 1901.

Couturat, 1903: Louis Couturat, *Opuscules et fragments inédits de Leibniz*. Paris, 1903.

On the nature and object of Russell's and Couturat's work on Leibniz, see Russell, pp. v-viii, 2-5, and *Mind*, N. S., Vol. XII, 1903, pp. 177-201.

Couturat made a profound study of Leibniz's published works and arrived independently at the same conclusion as Russell: that Leibniz's Metaphysics rests solely on the principles of his Logic. After this he extracted (and published in 1903) some of the most

interesting manuscripts of Leibniz preserved in the Royal Library of Hanover; and had in consequence to rewrite a large part of the book of 1901, but he did not have to modify his plan nor even to correct his chronological conjectures (Couturat, 1901, pp. x-xiv).

De Morgan's *Newton*: Augustus De Morgan, *Essays on the Life and Work of Newton.* Edited with Notes and Appendices by Philip E. B. Jourdain. Chicago and London, 1914.

G: C[arl] I[manuel] Gerhardt (Ed.), *Die philosophischen Schriften von G. W. Leibniz.* Berlin, 1875-1890.

G., 1846: C. I. Gerhardt (Ed.), *Historia et Origo Calculi Differentialis a G. G. Leibnitio conscripta. Zur zweiten Säcularfeier des Leibnizischen Geburtstages aus den Handschriften der Königlichen Bibliothek zu Hannover.* Hanover, 1846.

G., 1848: C. I. Gerhardt, *Die Entdeckung der Differentialrechnung durch Leibniz, mit Benutzung der Leibnizischen Manuscripte auf der Königlichen Bibliothek zu Hannover.* Halle, 1848.

G., 1855: C. I. Gerhardt, *Die Geschichte der höheren Analysis. Erste Abtheilung* [the only one which appeared]: *Die Entdeckung der höheren Analysis.* Halle, 1855.

G. *Bw.*: C. I. Gerhardt (Ed.), *Der Briefwechsel von Gottfried Wilhelm Leibniz mit Mathematikern.* Vol. I, Berlin, 1899. Cf. De Morgan's *Newton*, p. 106.

G. *math.*: C. I. Gerhardt (Ed.), *Leibnizens mathematische Schriften.* Berlin and Halle, 1849-1863. See De Morgan's *Newton*, pp. 71-72.

Guhrauer: G. E. Guhrauer, *Gottfried Wilhelm Freiherr von Leibnitz: Eine Biographie.* 2 vols. Breslau, 1846.

Klopp: Onno Klopp (Ed.), *Die Werke von Leibniz.* Hanover, 1864-1877.

Latta: Robert Latta (Tr.), *Leibniz:The Monadology and other Philosophical Writings.* Translated, with Introduction and Notes. Oxford, 1898.

Merz: John Theodore Merz, *Leibniz.* No. 8 of Blackwood's "Philosophical Classics for English Readers." Edinburgh and London, 1907.

Montgomery: George R. Montgomery, *Leibniz: Discourse on Metaphysics, Correspondence with Arnauld, and Monadology.* Chicago and London, 1902.

Rosenberger: Ferdinand Rosenberger, *Isaac Newton und seine physikalischen Principien. Ein Hauptstück aus der Entwickelungsgeschichte der modernen Physik.* Leipsic, 1895.

In Rosenberger's book, the passages which are relevant to Leibniz's work are as follows: Leibniz's mathematical correspondence with Oldenburg from 1674 (series for area of circle), Collins, and Newton (pp. 439-448); a short note on Leibniz's manuscripts (p. 447); Leibniz's publications of 1684 and 1686 (pp. 448-450); the progress of the calculus in the hands of Leibniz, the Bernoullis, and others (pp. 455-460); and the events which led up to the controversy and the controversy itself (pp. 460-506). Besides this, Leibniz's physical views, and so on, are mentioned on pp. 231-234, 239-247, 411-412, 512, 514-520.

Russell: Bertrand Russell, *A Critical Exposition of the Philosophy of Leibniz, with an Appendix of Leading Passages.* Cambridge, 1900.

Sorley: W. R. Sorley, "Leibnitz", *Encyclopaedia Britannica*, 9th ed., Vol. XIV, pp. 417-423. Edinburgh, 1882.

The same writer's article on Leibniz in the latest (11th) ed. of this *Encyclopædia* (Vol. XVI, Cambridge, 1911, pp. 385-390) is almost a reproduction of the above article: the body of the article has been somewhat condensed and the Bibliography at the end expanded.

Fren: A. Trendelenburg, *Historische Beiträge zur Philosophie.* 3 vols. Berlin, 1867.

U: Friedrich Ueberweg, *System der Logik und Geschichte der logischen Lehren.* 3d ed. Bonn, 1868.

LEIBNIZ'S LIFE AND WORK.

GOTTFRIED WILHELM LEIBNIZ was born on June 21, 1646, at Leipsic. His father and mother both belonged to what we may call the learned classes, and the Leibniz family had been known for some generations. The father of the philosopher was a notary and a professor of philosophy in the University of Leipsic. He had been married three times, Gottfried Wilhelm, born when his father was forty-nine, being the only son of his third wife. When Leibniz was six years old his father died, and his education during his school years was directed by his mother. In his autobiographical memoir he mentions the various obscure studies in which he seems to have delighted at an early age. He entered the University of Leipsic in 1661 as a student of law, having already read much in the classics and in scholastic philosophy. The title of his dissertation for the bachelor's degree (1663), *De principio individui,* marks his connection with the thought of Ockham and Nicholas de Cusa. He was apparently also affected by Raimundus Lullus, in his conception of symbolic logic and calculation. Owing to the officialism of those who granted degrees, Leibniz was unable to conclude his academic career at Leipsic and he therefore left his native Saxony never to return. Eventually he was made Doctor of Laws at Altorf near Nuremberg.

In this later period he seems to have come under the influence of Renaissance thought as it was in Bacon and

Hobbes, and he was affected by the mathematical conceptions of Descartes. His desire to know everything that he could led him to communicate with the Rosicrucians of Nuremberg, and in connection with them he dabbled in their form of chemistry which seems to have been a mixture of magic and learned jargon. But more important than this introduction to physical science was the meeting of Leibniz with the Baron von Boineburg, who had himself some interest in alchemy. The Baron induced Leibniz to leave Nuremberg with him for Frankfort, and there he was made a councillor of the supreme court of the Elector. From this time on Leibniz lived among courtiers and jurists.

It was at this period that he began his writings on jurisprudence, which he conceived should be systematized and made logical. He also began his philosophic writing with two tractates on motion, and at the request of his patron he brought out with an introduction an edition of a work by Marius Nizolius which is an attack, largely formal, upon the scholastics. The philosophical development of Leibniz will, however, be dealt with elsewhere, and here we shall confine attention to his more public activities.

The European situation at the end of the seventeenth century was unstable, owing in great part to the diplomatic device of the balance of power. Louis XIV loomed large, especially upon the German horizon and he appears to have been chiefly moved by that peculiar Renaissance myth—glory. After various pursuits of this intangible goal his activities so alarmed the Duke of Lorraine that in July 1670, the Duke attempted to form a league with the Electors of Mainz and Treves. It was suggested that England, Sweden and Holland should join the German states to prevent Louis from pursuing glory upon the banks of the Rhine. Leibniz was able to assist Boineburg in the nego-

tiations and he seems to have suggested a purely German league for defense against the military ambitions of France. It came to nothing. In the late summer, the duchy of Lorraine and the bishoprics were attacked and conquered by Louis, a beginning of evil still unended. Leibniz continued to urge the union of the German princes.

In 1672 he accompanied Boineburg to Paris, nominally upon private business of the Baron's, but in reality to attempt to turn the attention of the French king away from Germany and Holland. Leibniz had already worked out a scheme, which indeed had been suggested before, of an invasion of Egypt by Christian troops under the leadership of the French king. Glory, he conceived, might be there; and in any case Europe would be left in peace. The scheme was actually presented to and acknowledged by the foreign minister of Louis XIV, but nothing more was done in the matter. England and France attacked Holland—historians probably know why. At Paris, however, Leibniz continued, superintending the slow wits of the Baron's son and meeting various men of note and learning. At this time he seems first to have seriously studied mathematics and to have gone into the detail of the Cartesian philosophy.

From Paris he went with the Elector's ambassor for a short visit to London (January, 1673). The purpose of the embassy was to persuade Charles II to allow the interests of Germany to be considered in the treaty of peace with Holland. The request was refused, as it had been by Louis. But Leibniz took advantage of his visit to meet various learned men; and he was made a member of the Royal Society. We now hear for the first time of the work of Leibniz upon the higher mathematics. From 1675 to 1677 he was again in Paris and in 1676 completed his discovery of the differential calculus. Therein lay matter for controversy with Newton at a later date, but as it hardly

seems to be important which first made the discovery we may here avoid the issue. What is more interesting to remember is that Leibniz lived in London and Paris in the world of Christopher Wren and Robert Boyle, of Molière and Racine. There was a certain intellectual energy in the air which could not at that time be equaled anywhere else in the world.

In 1677 Leibniz left Paris. He had at one time thought of making his home there among the learned and the cultured, but an offer of a post in Hanover changed his plans. He visited London again for a week and then went on to Amsterdam and the Hague, where he met and conversed with Spinoza, and so to Hanover. For ten years he lived there as ducal librarian, and there he took up the task of collecting materials for a history of the house of Brunswick. But he was not to live retired. In the first place the European situation was again unsettled by the attack of Louis XIV upon Germany, in deliberate violation of a truce, on the obviously insincere plea that the Emperor was about to make peace with the Turks and might then turn his arms against France. The best defensive was known even then to be an offensive. The Revolution of 1688 in England gave new importance to the house of Hanover. Europe was thus already divided into Catholic and Protestant powers, which made utterly impossible the scheme of Leibniz and others for religious reunion.

From 1687 to 1691 Leibniz traveled to collect materials for his history in various parts of Germany and in Italy. He visited Venice, and at Rome was welcomed by various learned societies. There also he met learned Jesuits and heard of the missions in China, where he was given to understand there was much learning.[1] He paid a short visit to Naples and in 1689 reached Modena. But the new

[1] He suggests sarcastically in his letters that as the Europeans were sending missionaries to China to teach the truths of revelation, the Chinese should send missionaries to Europe to teach us the practice of natural religion.

stage in his life is marked chiefly by his connection with Berlin. He became what was practically a diplomatic agent there in 1700, and he wrote various political essays in support of Austria and of the making of Prussia into a kingdom. In Berlin also Leibniz met Christian Wolf, with whom he continued a correspondence from 1704 until his death, and who was recognized later as his philosophical successor.

We have an account of his personal appearance at about this date left by his secretary. He is said to have been a small man with broad shoulders and a slight stoop. His eyes were keen but small; his hair was originally dark but he had lost it all, and on his bald head there stood a bump the size of a pigeon's egg. It was, however, an age of wigs. His habits were ascetic. He slept little, and often in his chair without attempting to go to bed. He would go on with his reading even when suffering from an occasional illness. His emotional adventures were few, if at least we can judge from the fact that when he was fifty he proposed marriage to a lady who took time to consider it, whereupon Leibniz seized the opportunity to reconsider.

In public work the activity of Leibniz was of two kinds, diplomatic or juristic and academic. He conceived the idea of a logical jurisprudence, and his early attention seems to have been fixed upon the political situation. In 1659 he wrote an essay on the election of the king of the Poles, and in 1667 a *Nova methodus discendae docendaeque Jurisprudentiae*. His chief purpose, however, was exactness of definition and systematic treatment, and although he served in public life as a learned jurist and diplomat it is not in this sphere that he has contributed most.

Another public activity was his devotion to the religious reunion of Christendom. His attempts to reunite the Christian churches arose partly from his own training

and sentiments, partly no doubt from the fact that he was librarian at Hanover under the Catholic duke and under his successor the Protestant Ernst August. It was hardly a hundred years since the Reformation was established in the north and men of good will still shrank from taking it for granted that there must be divergence of religious forms and beliefs in Europe. Leibniz knew the scholastics and the best of the older Catholicism. He saw and appreciated the contemporary work of the Jesuits and he lived in the midst of a society very varied in its religion. Therefore he joined with enthusiasm those who hoped for some compromise between church officials and theologians of the old and the new schools. Most of his work was done by correspondence. On this subject he wrote to many Catholics, but the most important of his letters were addressed to Bossuet. The courtier bishop and theologian set out with great clearness the claims of the See of Rome. He said that Protestants were opinionated, that there was no evidence for Rome's ever having treated heretics as equals, and that the decrees of the Council of Trent could quite reasonably be accepted. Bossuet broke off the correspondence in 1694; but it was renewed and finally broken off by Leibniz in 1701. They could not agree, among other things, as to whether the Council of Trent should have introduced the Apocrypha into the Biblical canon.

Feeling ran fairly high even in the correspondence of scholars, although theological emotions had somewhat subsided since the days when the fathers of the Council of Trent pulled out each other's beards in an agony of excitement as to whether there was justification by faith only. Leibniz saw that the hope of any compromise grew less as each form of religion was more rigidly institutionalized, and doubtless those on the other side saw that the new churches lacked none of the assurance of the old. There was the added difficulty of political division more or less

[redacted]

of such brilliant accidents as Bacon and Hobbes or, in Leibniz's day, Newton, the universities were stiff with formulas. The religious orders in the Catholic countries were wealthy and their members had abundant leisure, but they had forgotten the possible connection of intelligence with religion. The older orders contained only commentators on the great scholastics, and the view taken of their duty to humanity was narrow and antiquated.

In Italy the custom had begun of cooperation between investigators, free from the traditions and the tutorial burdens—lightly borne indeed—of the universities. This is the origin of academies. They are the signs of the Renaissance, as universities are of the Middle Ages. They belong to the period of the humanists and polymaths and they lived on the appetite for new things which was only hampered by the mutual jealousy of their members. The Royal Society of London had been founded in 1660, the Paris Académie des Sciences in 1666; and it is with these two that Leibniz is chiefly connected. From his experience of their utility, he seems to have come to the conclusion that the idea of academies was valuable. Its importance for us here is largely historical, for academies have become, as universities had in Leibniz's day, opportunities for the mutual admiration of the obsolete. Their purpose, at least in the public mind, is rather to register the approval of established authorities than to give opportunity for new and fruitful departures from tradition. It is all the more important to recognize that they were once revolutionary intellectual associations, and it is as such that Leibniz looked to their principles as full of promise for the development of civilization.

Academies mark the new age in learning in two ways: In the first place an academy is a free association for investigation and the application of science to every-day needs and not for teaching or for explaining tradition.

This is one example of the mood of the Renaissance. The value set upon exceptional ability and the impulse to individual exploration in the intellectual as well as in the geographical world are here embodied. The famous Florentine Academy and the Roman Society which had an unfortunate notoriety under Paul II, were to their members, as they were to the public, associations of those who were willing and able to go beyond the known bounds of human knowledge. And the same spirit, less "pagan" on the one hand but more scientific on the other, was to be found in France and England during the late seventeenth century. The immense promise of the future gave the academies their best energy, and this promise could only be realized, it was felt, by individual or associated investigation into nature. Nothing could be more different from the spirit in which the universities had been founded: and in this spirit of progressive thought we have made but little advance upon the Renaissance enthusiasm.

In another sense the academies of Leibniz's day may be recognized as belonging to a stage of intellectual progress which has now been passed. We have seen that they are for the exceptional, by comparison to the universities. But on the other hand the Renaissance, even as late as the seventeenth century, was a period in which civilization depended upon a small clique in a world of uneducated and half brutalized "workers." Perhaps that world has not altogether disappeared. The position, however, of Descartes, Leibniz, and most scholars or scientists of the seventeenth century, could hardly be paralleled in our days. It is the position of courtiers, dependents and hangers-on of "great" men. Academies, indeed, still preserve the memory of their dependence upon favor as universities still preserve their old connection with the clergy. But we should be doing the activity of Leibniz an injustice if we did not allow for the limitations within which he worked.

The "reading public" was small, and the centers of civilization few. In addition to the London and Paris of his day we have a world-wide connection of great cities, and in place of his unwashed and semi-educated patrons we have vast numbers of men and women quite capable of appreciating a new scientific or literary idea. His achievement must, therefore, be measured by reference to the slender resources at his disposal, and we must imagine him rather a pioneer in the work of civilizing humanity than an exponent of all that may be done in that high task.

Leibniz was introduced to the Royal Society as a member in 1673; and he began his communications with the Paris Académie in 1675; though he could not become a member, as he was a Protestant. Both societies were looked upon as the very latest thing in learning and their members were often laughed at for their fantastic ideas. Swift's *Gulliver* and Butler's *Hudibras* contain the contemporary popular view of the practical applications of this new science.

Such was the institutional organization of learning. On the other hand, knowledge had vastly increased since the universities arose and was still increasing too quickly for the academies to assimilate or systematize it. We must, indeed, allow that there was much in the material valued by the academies which has eventually turned out worthless, although it is from what they collected that the most valuable part of our science arose. We must imagine a time when scholars spent as much time in devising a machine for making calculations as in elaborating the new mathematics. Out of such facts come the enthusiasm of Leibniz for organized learning. And this does not make him simply a passive agent of the vague needs of his time, for it required no little insight to grasp the situation and to suggest an advance.

The first need which appealed to Leibniz was that of systematization. He was himself, as we have seen, vastly learned, and he was also one of those few men whose reasoning had not been overcome by his learning. He was master of his "subjects," not they of him, and the much he had only gave him an appetite for more. But before his eyes there stretched the unlimited details of acquired learning then possessed by the scholars and the illimitable vistas of possible increase. He must have felt, first, like that librarian of Anatole France who pulled down upon himself his own catalog and died of it. And next, in the jungle of "facts" he felt himself helpless even to utilize what he knew was there. "We are poor," he writes, "in the midst of riches, and we are hampered by the excess of our resources." The primary need, therefore, was a system of the sciences. Of this there are two renderings in Leibniz, belonging to different stages in the development of his own conceptions. The former begins with theological and moral science and hardly includes what we should call physical science. The second plan gives theology and morality a much more restricted space and is chiefly concerned with what we should call science and history. This marks the change of emphasis in Leibniz's mind [illegible] moved more and more towards mathematics [illegible] methods of thought.

During all this time Leibniz was atte[illegible] lish some exact and universally valid sy[illegible] tion in philosophy such as was alread[illegible] gebra. This would be, as he conti[illegible] of Ariadne in the labyrinth of a[illegible] hoped, as most men did then, [illegible] philosophy. But we may put [illegible] ment when the relations of [illegible] considered. What is impo[illegible]

who know do not rule,
those whose only use
silently their primitive
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sought and more easily
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in France and England should be assisted

have contrived the foundation of another
resden but for the war with Charles XII
Vienna, Leibniz tried from 1712 to 1714
ort of the Emperor for an academy. He
the possibility of its depending, according
plan, upon the subscriptions of its members,
slight subvention from the funds for hospitals,
ing a Pro tant, he had to declare that he did not
to be pre of the proposed society; but no con-
spicion and even the open hostil-
re strong enough to prevent the

were as irresponsive as the princes, and Leibniz's ideal society was never founded.

It is worth while for us, however, to remember his plan. He had been much impressed by the religious orders in Paris and especially by the Jesuits. They had riches and organization and they worked independently of local or national interests for the "eternal welfare" of men. Why should there not then be, said Leibniz, an order of the intelligent and learned, "in which besides religion the happiness of men in the present world should be arrived at?"[3] Such a society would be "philadelphic," and could not be founded except with some religious enthusiasm:[4] but it would have the devotedness and the organization of the Society of Jesus, without the rigidity of rule and the concentration upon authority. It would be an *Internationale des Savants,*—a spiritual power. Its members would preserve and increase our knowledge of the secrets of nature and they would study and publish knowledge of public utility.

The various appliances which might be invented are hinted at in Leibniz's attempt to make a machine for pumping the water out of mines, and another for controlling fire. He proposed the conservation of forests, the institution of a metric system of weights and measures, and various other practical reforms. His conception of the society of Wise Men is like that of Bacon's college in the New Atlantis. There was here the common Renaissance forecast of the elaborate machinery we have now at our disposal. But Leibniz perceived that unless an international society with humanitarian interest were devoted to this purpose, the growth of knowledge would be retarded and in its practical applications it would be enslaved to the prejudices and pettinesses of local lords or rival factions.

[3] Cf. letter quoted in Couturat, 1901, p. 507, note 3.

[4] "Societatis talis stabiliri nulla melius ratione posset quam religionis conspiratione."

And so indeed it has been. Those who know do not rule, but their knowledge is controlled by those whose only use for "science" is to attain more violently their primitive purposes. Leibniz foresaw what we know, that explosives and engines of destruction are first sought and more easily made effective than contrivances for making labor lighter or life more pleasant. We still use the houses of his century but we have discarded its guns as unworthy of us. The spiritual power is still longed for by the French political theorists. The Internationale was never founded.

But Leibniz's ideals were not altogether without practical effect. He saw with regret that Germany was without any society such as the Royal Society or the Académie des Sciences. He therefore suggested an academy at Berlin, pointing out both the practical utility of such a society and the prestige it would give. For nine years he worked at making the authorities accept the idea, and the Berlin Academy was at last established in 1711 (Jan. 19). Leibniz's work was a direct evidence of the dependence of the civilization of one country upon the advance made in others. It was not simply as a rival that the Berlin Academy was brought into existence but in order that the progress initiated in France and England should be assisted in Germany.

He would have contrived the foundation of another such society at Dresden but for the war with Charles XII of Sweden. At Vienna, Leibniz tried from 1712 to 1714 to obtain the support of the Emperor for an academy. He even suggested the possibility of its depending, according to the English plan, upon the subscriptions of its members, with some slight subvention from the funds for hospitals, etc. Being a Protestant, he had to declare that he did not desire to be president of the proposed society; but no concession could buy off the suspicion and even the open hostility of the Jesuits, who were strong enough to prevent the

Academy of Vienna from being founded. Leibniz, however, continued for some years to reside in Vienna, and his influence at least brought some recognition for unecclesiastical learning. He was able also at this period to affect the new civilization of Russia.

Leibniz had met Czar Peter at Hanover in July, 1697. The Czar had come, practically in disguise, as a member of his own embassy, and he was evidently open to new ideas. In 1708 Leibniz suggested to him the formation of a scientific society in Russia; but the war with the Turks prevented any action being taken.[5] Leibniz, as usual, made a note of the subjects to which such a society, in view of its surroundings, could specially devote itself. He suggested that geography would be most naturally the chief task of a Russian society, considering the vast unknown upon which Russia bordered. Thus in his mind there was an intimate connection between the foundation of national academies and the special work of each for the general good of all men.

So far we have seen how Leibniz suggested a religious or humanitarian task to be adopted by established societies, and then urged with partial success the formation of different new local or national societies. But he had all along kept before him the ideal of an international union of men "of learning and of good will." Thus in May, 1696, he wrote to Placcius: "Nothing is more useful than the union of the learned in societies. It would be best that there should be one such universal society divided as it were into distinct colleges. For such is the connection between the different parts of knowledge that only by mutual friendliness and assistance can they be made to progress." And again in October, 1697, he wrote: "So long as something valuable is done, I do not care whether it be done in Germany or in France, for I desire the good of the whole

[5] A society was, however, founded in 1724 at St. Petersburg.

human race. I am not a lover of Greece or Rome but of man."[6]

It is sufficiently obvious, then, that Leibniz although an active supporter of scientific progress in different countries, was a convinced internationalist. He does not conceive the two attitudes to be inconsistent, since in every step forward made by separate nations he saw a promise of good for the whole human race. But events since Leibniz's day have gradually obscured the more comprehensive ideal, and the primitive jealousies of different racial groups have taken control of science and even of the resources of art. Progress has been more rapid in those applications of science which divide men from one another. History and literature have become in every nation an apologia or a panegyric of that nation. The current of events was directed not by the plan of idealists but by the appetite of princes. Leibniz himself was not unaffected. In 1707 he was sent on a secret mission to Charles XII of Sweden who was at that time pursuing glory in Auerstadt near Leipsic; and Leibniz's scholarship was used by the Emperor for political writing about the situation following the Peace of Utrecht.

During all this time Leibniz had been continuing his official work upon the chronicles of the house of Brunswick. He speaks of the *mille distractions* of his life, which kept him from philosophy, and he complains to his friends that "at a court nothing like philosophy is wanted or asked for."[7] He had, however, written the *Nouveaux Essais* in 1704, and in 1710 the *Theodicée*. From 1711 until 1714 he lived chiefly at Vienna and there, in about 1712, he was made an imperial privy councillor and a baron. The *Monadologie* was written in 1714, to be presented to Prince Eugene: and when Leibniz returned to Hanover in that

[6] "Je suis non pas φιλέλλην ou φιλορωμαῖος, mais φιλάνθρωπος."

[7] Letter to Placcius, 1695, "in aulis scis longe aliud quaeri atque exspectari."

year he found that the Elector had left, owing to the death of Queen Anne in England. Leibniz had hopes of following his patron to London, and had in fact thought some years before that he would find in London more congenial companionship than in Hanover. But the Elector, now king of England, told the philosopher to go on with his writing of the history of the house of Brunswick. Unfortunately Leibniz had expressed his opinion some time before that the customs of the English should not be interfered with by their king, and the Hanoverian ministers viewed his possible liberalism as a danger. A legend says that George I was proud of having a Leibniz in one of his dominions and a Newton in the other; at any rate he kept them apart.

A form of arthritis, from which Leibniz had suffered for some years and to which his sedentary habits contributed, became more acute in 1715. The history of the house of Brunswick was, however, prepared in that year for publication.

Leibniz died on November 14, 1716. At his deathbed no clergy attended as he had seldom or never been to church, and no one but his secretary followed his body to the grave. The court was aware of the little value set by George I upon a mere historian of his greatness. No notice was taken of his death, even by the learned, except that a decorative oration was pronounced upon him in the Académie des Sciences of Paris. Berlin and London made no sign.

The impression made by Leibniz on his contemporaries seems not to have been very great, or it may be that the unfortunate controversy with Newton prevented his being judged rightly by the English scholars who would perhaps best have appreciated his work. Upon his own countrymen the work of Leibniz made no impression for many years after his death. He lived among courtiers and de-

pended for his livelihood upon what could be spared by princes after their expenditure in the pursuit of warlike glory. He suffered for his security. The state in the years since the Middle Ages has taken credit to itself for supporting art and science, as the church in earlier times is supposed to have done. The evidence for each claim is equally lacking.

In general character Leibniz seems to have been pleasant and not striking. In intellectual interest he is the representative of the old tradition of omniscient humanists who intervened between medieval scholasticism and modern thought. Devoutly religious in the untheological sense, he endeavored always to keep hold of the tradition of those who believed in the goodness of God. He does not seem to have experienced the heights or the depths of emotion, although he greatly valued Plato. But it is no small credit to his genius that he was able to see so keenly into the nature of things through the elegancies and sentimental egoisms of court life. And his human sympathy was far-seeing and comprehensive.

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THE LOGICAL WORK OF LEIBNIZ.

WHEN Leibniz's work is studied as a whole, as some of his remarks clearly show us that it ought to be studied,[1] we can see that his philosophy and his mathematics were founded in his logic. Although many have noticed the close connection of Leibniz's notions of his infinitesimal calculus and his monads, for example,[2] it was reserved for modern investigation to trace the complete story, both by reconstruction of Leibniz's thought and by taking into account hitherto unpublished documents written by Leibniz himself. This being the case, it is difficult not to complain of the way in which Leibniz's works have been published. Thus, Gerhardt, the editor of the most modern and complete collection of Leibniz's works, separated these works into "philosophical" and "mathematical." And yet Leibniz himself, in a letter to de l'Hôpital of December 27, 1694, had said: "My metaphysics is wholly mathematical";[3] and to Malebranche in March, 1699, he had said that "mathematicians need to be philosophers just as much as philosophers need to be mathematicians."[4]

In this article an attempt will be made to give an idea of Leibniz's logical work and plans for logical work. Great use has been made of Couturat's splendid book of 1901 mentioned in the Bibliography given above, but on some important points Couturat's account is supplemented. For example, this is so in the account (§ II) of the early ap-

[1] Cf. Couturat, 1901, pp. vii-ix.
[2] Thus cf. Latta, pp. 74-86.
[3] G. *math.*, Vol. II, p. 258.
[4] G., Vol. I, p. 356.

pearance of Leibniz's doctrine that the principle of identity held a very fundamental place in logic; in the sections (§§ IV, V) on the influence which guided Leibniz to a study of mathematics and on his mathematical work down to about the end of 1676; in the account (§ X) of the principle of continuity and its later developments; and in numerous footnotes throughout the paper. It cannot be too strongly emphasized that only these supplements are here treated at length, and that a knowledge of Couturat's book is assumed,—merely a tolerably full account of its contents has been given in the sections devoted to it.

I.

In a philosophical essay which Leibniz wrote in later life, under the name of "Gulielmus Pacidius," he said that when in his tenth year the library of his father, who was then dead, was thrown open to him, he seemed to be guided by the "*Tolle, lege*" of a higher voice, so that his natural thirst for knowledge led him to study the ancients and imbibe their spirit. "I burned," said he, "to get sight of the ancients, most of them known to me only by name, Cicero, Seneca, Pliny, Herodotus, Xenophon, Plato, and the historical writers, and many church fathers, Latin and Greek"; and soon it was with him as with "men walking in the sun, whose faces are browned without their knowing it."

It was characteristic of him to find some good in all he read.[5] "Like Socrates," he said, "I am always ready to learn";[6] and it was the study and spirit just mentioned, says Merz,[7] that led him to aim at two things which seemed to him to be foreign to the writers of the day: in words to attain clearness, and in matter usefulness.[8] The first aim led him to the study of logic, and, before he reached the age of twelve[9] he plunged with delight into the study

[5] G., Vol. VII, p. 526; Latta, pp. 1-2; Russell, pp. 5-7.
[6] Latta, p. 17.
[7] Merz, p. 13.
[8] *In verbis claritas, in rebus usus.*
[9] Couturat, 1901, pp. 33-34.

of scholastic logic.[10] He wrote out criticisms and plans for reform, and confessed that in later life he found great pleasure in re-reading his rough drafts written at the age of fourteen. At this age the idea occurred to him that just as the "predicaments" or categories of Aristotle serve to classify simple terms (concepts) in the order in which they furnish the matter of propositions, complex terms (propositions) might be classified in the order in which they furnish the matter of syllogisms, or of deduction generally. Neither he nor—probably—his teachers knew that that is exactly what geometricians do when they arrange their theorems in the order in which they are deduced from one another. Thus it was the mathematical method which was Leibniz's logical ideal even before he knew it, and it is not surprising that later on he took it as model and guide and grew to regard logic as a "universal mathematics."

Leibniz continued[11] to meditate on his idea of a classification of judgments about which his teachers had given him no information that was to the point, and it seems to have been in his eighteenth year that he arrived at thinking that all truths can be deduced from a small number of simple truths by analysis of the notions which are contained in them, and that all ideas can be reduced by decomposition to a small number of primitive and indefinable ideas. Thus we would only have to enumerate completely these simple ideas and thus form an "alphabet of human thoughts," and then combine them together, to obtain successively all complex ideas by an infallible process. This idea was a great joy to Leibniz, and while as a student of law at Leipsic University he was writing a dissertation on the necessity of introducing philosophical principles and reasoning into matters of law, and maintaining that the ancient jurists had brought so much thought and knowledge to bear upon their

[10] A short and good summary of the classical or syllogistic logic is given at *ibid.*, pp. 443-456.

[11] *Ibid.*, pp. 34-35.

subject that the principal task which they left to their successors was the systematic arrangement of the matter which they had collected, he was composing his treatise *De arte combinatoria*.[12] In it he showed that one of the principal applications of the art of combinations is logic, and more particularly the logic of discovery as opposed to demonstrative or syllogistic logic. The fundamental problem of the logic of discovery is, Given a concept as subject or predicate, to find all the proportions in which it occurs. Now, a proposition is a combination of two terms, a subject and a predicate. Thus the problem reduces to the problem of combinations.

In the latter part of this dissertation, Leibniz used and criticized the ideas of his predecessors, Raymond Lulle and others;[13] and one of the first applications given of the art of combinations was to the determination of the number of moods of the categorical syllogism.[14] Here I will draw attention to a relevant extract from a rather important manuscript of Leibniz. It was not referred to by Couturat, but is translated as the second of Leibniz's manuscripts on the infinitesimal calculus given in another article in this number of *The Monist*.

II.

When speaking of his early logical studies, Leibniz said in his *Historia et Origo*:[15] "When still a boy, when studying logic, he perceived that the ultimate analysis of truths that depend on reason reduces to these two things: definitions and identical truths; and that they alone of essentials are primitive and indemonstrable. And when it was objected to him that identical truths are useless and nugatory,

[12] Couturat, 1901, pp. 35-36; cf. also Merz, pp. 17-18, 106-114; Cantor, Vol. III, pp. 41-45.

[13] Couturat, 1901, pp. 36-39.

[14] On this and on Leibniz's later work on the syllogism, see *ibid.*, pp. 2-32.

[15] Cf. G., 1846, p. 4.

he showed the contrary by illustrations. Among other illustrations he showed that the great axiom that the whole is greater than the part could be demonstrated by a syllogism whose major premise was a definition and whose minor premise was an identical proposition. For, if of two things one is equal to a part of the other, the former is called the *less* and the latter the *greater*; let this be taken as the definition. Now, if to this definition we add the identical and undemonstrable axiom that everything possessed of magnitude is equal to itself, or A = A, then we have the syllogism:

"Whatever is equal to a part of another is less than that other (by definition);

"But the part is equal to a part of the whole (namely to itself, by identity);

"Therefore the part is less than the whole, Q. E. D."[16]

In another draft of the *Historia et Origo,* Leibniz speaks more at length about these early logical studies:[17] "Hitherto, while still a student, he was striving to bring logic itself to a certitude equal to that of arithmetic. He had observed from the first figure it was possible that the second and third might be deduced, not by employing conversion (which indeed itself seemed to him to need proof) but by employing solely the principle of contradiction; moreover, that conversions themselves could be demonstrated by the aid of the second and third figures by employing identical propositions; and, lastly, conversion being now demonstrated, by its aid the fourth figure could also be demonstrated; and thus that it was more indirect than the former (figures). Also he wondered very much at the force of these *identical truths,* for they were commonly considered to be nugatory and useless.[18] But later he perceived that the whole of arithmetic and geometry arose from identical

[16] Cf. Couturat, 1901, pp. 204-205. See also below, p. 590.
[17] G., 1846, p. 26, note 17.
[18] Cf. Couturat, 1901, pp. 8-12.

truths; and that, in general, all truths that were indemonstrable, if depending on pure reasoning, were identical; and that these combined with definitions produce identical truths. He gave an elegant example of this analysis in a demonstration of the theorem that the whole is greater than its part."

To Couturat's words[19] that Leibniz was concerned with showing the utility of identical propositions in reasoning and with defending them against the reproaches of insignificance and sterility urged against them by the empirical logicians, we may add two things: First, Leibniz seems to have held from early days the opinion that the foundations of logic are definitions and identical axioms;[20] secondly, in the *Historia* just mentioned, he traces to an identity his earliest mathematical discoveries in the summaton of series.

III.

We will now return to Leibniz's application of the art of combinations to the logic of discovery.[21] On analyzing all concepts by defining them—that is to say, by reducing them to combinations of simpler concepts— we arrive at a certain number of absolutely simple and indefinable concepts, and these "terms of the first order" are denoted by some such signs as numerals. "Terms of the second order" are obtained by combining in pairs those of the first order; and so on for terms of higher orders. Leibniz represented a compound term by the (symbolic) product of the numbers corresponding to the simple terms.

Leibniz was at that time still a novice in mathematics, and that explains many of the imperfections of the dissertation on the art of combinations; but still this early work

[19] *Ibid.*, p. 12.

[20] G., Vol. V, p. 92; Russell, pp. 17-19, 169; Couturat, 1901, p. 203. Cf. also the analogous example quoted from Leibniz and criticized by Frege, *Die Grundlagen der Arithmetik*, Breslau, 1884, pp. 7-8; Couturat, 1901, pp. 203, 205-207.

[21] Couturat, 1901, pp. 39-50.

contains the germ of his whole logic, which was with him a life-long study. That Leibniz was then a novice in mathematics comes out in the fact that he did not at first imagine his logic as a sort of algebra, but, since he was probably influenced by contemporary schemes, as a universal language or script.[22] This he had mentioned in his dissertation of 1666, and he developed it in the following years, especially from 1671 onward.[23] His "rational script" was, he says, a most powerful instrument of reason, and that it would promote commerce between nations should be esteemed the least of its uses. The notations or "characters" of a "real characteristic" represents ideas immediately and not words for them; thus, Egyptian and Chinese hieroglyphics and the symbols used by the alchemists for denoting substances are "real characters," and so they can be read off in various tongues; and, further, the "rational language" is formed on philosophical principles and is a help to *reasoning*.

IV.

According to Gerhardt[24] and Couturat,[25] Leibniz was led by logical investigations to the study of mathematics. About[26] the middle of the seventeenth century the study of mathematics in the universities of Germany was in a very bad state; and it is possibly enough to mention that his teachers were Johann Kühn and Erhard Weigel at the universities of Leipsic and Jena respectively. Still, Weigel seems to have gained a certain respect from Leibniz, and to have influenced him.[27] However, the facts that Leibniz had entered into correspondence with such men as Otto von Guericke and the learned Jesuit Honoratus Fabri of

[22] *Ibid.*, pp. 51-80. [23] *Ibid.*, pp. 59-61. [24] G., 1848, p. 7; G., 1855, p. 53.
[25] *Op. cit.*, p. 279. This is of course based on Leibniz's own statements.
[26] For the rest of this section, cf. G., 1848, pp. 7-9; G., 1855, pp. 53-54.
[27] Cf. Latta, p. 3.

Rome, and had sent the two parts of his *Hypothesis physica nova* to the lately founded learned Societies at London and Paris, show that Leibniz's active spirit was by no means satisfied with the knowledge he obtained in his university career. Before 1671 he had to depend almost entirely on books which came by chance into his hands, and thus it was that he was only acquainted with the beginnings of mathematical science and was for the most part ignorant of the progress made by the French, British and Italians during the seventeenth century. Also we must remember that he then considered law and history as his life-studies and thus only studied mathematics rather by the way and without any special industry. However these studies were very important for Leibniz, for he always kept in view their connection with logical researches and thus obtained exercise in expressing concepts by general signs. His first mathematical and philosophical writing of 1666 bears this character, and Leibniz himself repeatedly referred to it in the controversy about the discovery of the calculus.

In a letter[28] written from Mainz in the autumn of 1671 to the Duke of Brunswick-Lüneburg Leibniz announced a list of discoveries and plans for discoveries, arrived at by means of this new logical art, in natural science, mathematics, mechanics, optics, hydrostatics, pneumatics, and nautical science, not to speak of new ideas in law, theology and politics. Among these discoveries was that of a machine for performing more complicated operations than that of Pascal—multiplying, dividing, and extracting roots, as well as adding and subtracting.[29]

For Leibniz's mathematical education his stay in Paris, where he went in March of 1672 on a political mission, is

[28] Klopp, Vol. III, pp. 253ff.

[29] Sorley, p. 419. In G., 1848, p. 17; Latta, p. 6; and Merz, p. 53, it is implied that this machine was invented at Paris. This was also implied by Leibniz himself in § I of the article below on Leibniz's manuscripts relating to the infinitesimal calculus; but see Couturat, 1901, pp. 295-296. On the machine, see Cantor, Vol. III, p. 37.

of the greatest importance. Here for the first time he came into contact with the most eminent men of science of the time, and especially with Huygens who had presided over the French Academy since the year 1666. When Huygens published his celebrated *Horologium oscillatorium*, he sent a copy to Leibniz as a present. Leibniz saw from this work how very ignorant he was of mathematics, and his ambition to excel in this science flared up. In scientific conversations with Huygens the properties of numbers came into discussion, and Huygens, perhaps to test the talent of his new pupil, proposed to him the problem of finding the sum of a decreasing series of fractions whose numerators are unity and whose denominators are the triangular numbers. Leibniz found the correct result.[30]

Leibniz's intercourse with Huygens was interrupted by a journey to London in January of 1673.[31] In London, just as in Paris, he sought out the acquaintanceship of the celebrated men of England who lived in the capital. He had been in correspondence since 1670 with Henry Oldenburg, the secretary of the Royal Society, and met the mathematician Pell at the house of the chemist Robert Boyle. The conversation turned on the properties of numbers and Leibniz mentioned that he possessed a method of summing series of numbers by the help of their differences. When he explained himself more fully about this, Pell remarked that the method was contained in a book of Mouton called *De diametris apparentibus Solis et Lunae*. Leibniz had hitherto not known of this work; he borrowed it at once from Oldenburg, turned over its pages, and found that Mouton had obtained the same result in another way, and that his own method was more general.[32] By Pell Leibniz's attention was drawn to Mercator's *Logarithmotechnia*,

[30] G., 1848, pp. 17-19; G., 1855, p. 54.

[31] Cf. Cantor, Vol. III, p. 30.

[32] See the letter of Leibniz of February 3, 1673, to Oldenburg (G. *math.*, Vol. I, pp. 24ff).

especially because of the quadrature of the equilateral hyperbola contained in it, and Leibniz took this work with him to Paris. After his return to Paris he began, under Huygens's guidance, the study of the whole of higher mathematics. The *Géométrie* of Descartes, which hitherto he had known only superficially, the *Synopsis geometrica* of Honoratus Fabri, the writings of Gregory St. Vincent, and the letters of Pascal on the cycloid, were his guides.[33]

V.

We have another and rather different version of the way in which Leibniz was led to the study of mathematics. It was when he began to study at Leipsic University, which he entered in 1661—his fifteenth year—that he first became acquainted with the modern thinkers who had revolutionized science and philosophy.[34] "I remember," said Leibniz, "walking alone, at the age of fifteen, in a wood near Leipsic called the Rosenthal, to deliberate whether I should retain the doctrine of substantial forms. At last mechanism triumphed and induced me to apply myself to mathematics."[35]

In a letter of 1669 to Jacob Thomasius, one of his former teachers of philosophy at the University of Leipsic, Leibniz contended that the mechanical explanation of nature by magnitude, figure and motion alone is not inconsistent with the doctrines of Aristotle's Physics, in which he found more truth than in the Meditations of Descartes. Yet these qualities of bodies, he argued in 1668, require an incorporeal principle for their ultimate explanation. In 1671 he issued a *Hypothesis physica nova,* in which,

[33] G., 1848, pp. 19-20; G., 1855, pp. 54-55. On Leibniz's mathematical work of about this time, see Cantor, Vol. III, pp. 76-84, 115-118, 161-168, 179-184, 187-189, 191-216, 320-321; G., 1848, p. 15; G., 1855, pp. 33, 37-38, 48; G. 1846, p. xii, and the manuscripts on the calculus translated below; and Merz, pp. 50, 54-62. On the subsequent controversies to which this work gave rise, see Merz, pp. 84-96, 94-99, and Vol. III of Cantor.

[34] Latta, pp. 2-3.

[35] Cf. Merz, pp. 14-15; Latta, p. 3.

agreeing with Descartes that corporeal phenomena should be explained from motion, he contended that the original of this motion is a fine ether which constitutes light and, by penetrating all bodies in the direction of the earth's axis, produces the phenomena of gravity, elasticity and so on. The first part of the essay on concrete motion was dedicated to the Royal Society of London; the second part, on abstract motion, to the French Academy.[36]

VI.

It was in 1676 that Leibniz[37] seems first to have dreamed of a language which should at the same time be a calculus or algebra of thought, and then he definitely borrowed from mathematics his logical ideal.

But he soon found[38] that the construction *a priori* of a "rational language" was not so simple as he had believed at first, and in 1678 set about a comparative study of living languages for the purpose of extracting and combining the simple ideas expressed in them and of founding a "rational grammar";[39] and this language was by no means to be a calculus.[40]

Leibniz's problems then were, first, to make an inventory of human knowledge in which all known truths were to be demonstrated by reducing them to simple and evident principles, and, secondly, to invent signs to express the primitive concepts and their combinations and relations.[41] The second part was called the problem of the "Universal Characteristic"[42]—the characters being both what he called "real" and useful for reasoning, like the signs of arithmetic and algebra,—and the first part that of the "demonstrative encyclopedia."[43]

[36] Sorley, p. 419. On Leibniz's view of nature as a mechanism and his philosophy, cf. also Merz, pp. 41-43, 67-68, 72-73, 137-190.

[37] Couturat, 1901, pp. 61-62.

[38] *Ibid.*, pp. 63-64.

[39] *Ibid.*, pp. 64-79.

[40] *Ibid.*, pp. 78-79.

[41] *Ibid.*, pp. 79-80.

[42] *Ibid.*, pp. 81-118.

[43] *Ibid.*, pp. 119-175.

It was Leibniz who seems to have been the first to point out explicitly that "a part of the secret of analysis consists in the characteristic, that is to say, in the art of making a good use of one's notations,"[44] and we know,[45] both from his great step in inventing a supremely good notation and calculus for differentials and integrals and from the way in which he spoke of it from the very first, that he had the philosopher's property of being conscious of the help given to analysis by the invention of a *calculus of mathematical operations*— not "quantities"—which was very analogous to the calculus of ordinary algebra. The accusations that Leibniz had stolen ideas for an infinitesimal method are not only mistaken but also irrelevant. Leibniz himself said, without much exaggeration, that all his mathematical discoveries arose merely from the fact that he succeeded in finding symbols which appropriately expressed quantities and their relations.[46] In this connection we may mention that from Leibniz's Characteristic proceeded, besides his infinitesimal calculus and his dyadic arithmetic,[47] the use of a certain numerical notation in algebra and especially in the solution of simultaneous algebraic equations, the analogy between the development of, say, a binomial expansion and the repeated differentiation of a product of two factors, so that integration may be regarded as the operation of differentiation with a negative exponent, and so on.[48]

VII.

Leibniz formulated the conditions of a good Characteristic,[49] and clearly realized that it forms the basis for an

[44] Letter of 1693; Couturat, 1901, p. 83.
[45] Cf. *ibid.*, pp. 83-87.
[46] G. *math.*, Vol. VII, p. 17; Couturat, 1901, p. 84; Russell, p. 283.
[47] This is considered in another article in the present number.
[48] Couturat, 1901, pp. 473-500; Cantor, Vol. III, pp. 110-112, 230.
[49] Couturat, 1901, pp. 87-89.

algebra of logic, a *calculus ratiocinator* in which the rules of reasoning are translated by laws like those of algebra, and reasoning becomes a machinelike calculating process which frees the imagination where its action is not essential and thus increases the power of the mind.[50] With this tendency to economy of thought we may, it would seem, connect the opinion which Leibniz held on the value of the reduction of geometrical reasoning to analysis. "What is best and most convenient," said he,[51] "about my new (infinitesimal) calculus is that it offers truths by a kind of analysis and without any effort of imagination, which often only succeeds by chance, and that it gives us over Archimedes all the advantages which Vieta and Descartes had given us over Apollonius."

VIII.

The elaboration of the encyclopedia presupposed the knowledge of a universal method which should be applicable to all sciences, a "general science."[52] Little by little, the great plan for the encyclopedia, which occupied Leibniz at intervals from his twentieth year up to the time of his death, gave place gradually to the more restricted project of "beginnings of the general science," in which Leibniz would have exposed the principle of his method, that is to say his whole logic—which was an art of discovering as well as one of judging and demonstrating. All deduction, so Leibniz contended, rests on definitions, identical propositions; and so all truths can be demonstrated except identical and empirical propositions.[53] A definition is "nominal" when it indicates certain distinctive characters of the thing defined, so as to permit us to distinguish it from any other;

[50] *Ibid.*, pp. 96-103. Cf. on this point Jourdain, *Quart. Journ. Math.*, Vol. XLI, pp. 324-325, 329-332. Cf. also Russell, pp. 170, 206-208, 283-284.

[51] *G. math.*, Vol. II, p. 104; Russell, p. 283.

[52] Couturat, 1901, pp. 176-282. Cf. Latta, pp. 206-207.

[53] Couturat, 1901, pp. 184-188.

but a definition is only "real" when it shows the possibility or the existence of the thing. Indeed, the geometrical method requires that we demonstrate the possibility or ideal existence of every one of the figures defined either by indicating its construction or otherwise, so that every definition implies a theorem.[54]

Since the thorough analysis of truths and notions is the ideal of science, it is important to demonstrate the axioms, that is to say, to reduce them to definitions and identical propositions.[55] Indeed, every truth, whether necessary or contingent, is a relation of logical inclusion which can be discovered by simple analysis of the terms.[56]

Another part of Leibniz's logic is formed by questions arising out of the calculus of probabilities: the logic of probabilities is the science of temporal and contingent truths, and was, for Leibniz, a natural complement of the logic of certitude. And with this are connected considerations on the method of the natural sciences and the art of discovery.[57]

This art of discovery was regarded by Leibniz as his greatest discovery. He had cultivated it from his youth; it was to penetrate its secrets that he studied mathematics, because the sciences grouped together under that name were then the only ones in which this art was known and applied; and it was by trying to perfect it that he made all his mathematical discoveries. Thus we see why Leibniz's logic, mathematics, and philosophy were so closely connected, and also why Leibniz tried to give to philosophy a mathematical form.[58] But to extend the mathematical method to all sciences, the very idea of mathematics must be generalized, and this generalization resulted in the "Uni-

[54] *Ibid.*, pp. 188-195. On this theory of definitions and Leibniz's classification of ideas, see *ibid.*, pp. 195-200.

[55] *Ibid.*, pp. 200-207.

[56] *Ibid.*, pp. 208-213. On other principles (sufficient reason, and so on), see *ibid.*, pp. 213-239.

[57] *Ibid.*, pp. 239-278.

[58] *Ibid.*, pp. 278-282.

versal Mathematics,"[59] whence arose a general logic of relations.[60] But the only algebra which Leibniz developed at all was what may be called attempts at a "logical calculus," dealing with the relations of identity and inclusion,[61] and the "geometrical calculus," dealing with the direct study of figures and spatial relations.[62] Both are particular applications of the Characteristic, and both are essays in Universal Mathematics.

We know now,[63] from Leibniz's manuscripts, that he possessed almost all the principles of the logic of Boole and Schröder, and on certain points he was further advanced than Boole. The chief reason why Boole succeeded where Leibniz failed is that Boole made the calculus of logic rest on the exclusive consideration of extension—and not intension—of concepts.

In criticism of the main points of Leibniz's logic Couturat[64] has advanced the following remarks. The postulates of Leibniz's logic are two in number: (1) All our ideas are compounded out of a small number of simple ideas; (2) Complex ideas proceed from these simple ideas by uniform and symbolical combination analogous to arithmetical multiplication. With regard to (1), the number of simple ideas is very much greater than Leibniz believed. With regard to (2), logical "multiplication" is not the only operation of which concepts are susceptible: we have to consider also logical "addition" and "negation." Leibniz, because he did not take account of negation, could not explain how simple ideas—which are all compatible with one another—can generate, by combination, mutually contradictory or exclusive complex ideas. Further, even if Leib-

[59] *Ibid.*, pp. 283-322. [60] *Ibid.*, pp. 300-318.

[61] *Ibid.*, pp. 323-387. These attempts began in 1679.

[62] *Ibid.*, pp. 388-430; Cantor, Vol. III, pp. 33-36; cf. also Couturat, 1901, pp. 529-538. A special article by Mr. A. E. Heath on the relation of Grassmann's ideas to Leibniz's will appear in the January issue.

[63] *Ibid.*, pp. 386-387. [64] *Ibid.*, pp. 431-441.

niz had succeeded in building up an algebra of classical logic, the logic of relations would still have remained outside. Leibniz was conscious of this and with him are to be found the first attempts at such a logic, but he did not go far, owing, it would seem, to an excessive respect for the authority of Aristotle.

We must always remember that, in his *Nouveaux essais*, Leibniz[65] laid stress on the importance of the invention of the form of syllogisms, and remarked that it is "a kind of universal mathematics whose importance is not sufficiently known"; and that he also remarked[66] that there are good asyllogistic conclusions, such as "Jesus Christ is God, therefore the mother of Jesus Christ is the mother of God," and "if David is the father of Solomon, without doubt Solomon is the son of David."

X.

We will now consider Leibniz's "law of continuity" and its later fortunes.

Leibniz, in the course of his letter of 1687 to Pierre Bayle on a general principle useful in the explanation of the laws of nature[67] says: "It [the principle] is absolutely necessary in geometry, but it succeeds also in physics, because the sovereign wisdom, which is the source of all things, acts as a perfect geometer, following a harmony to which nothing can be added.... It may be enunciated thus: 'When the difference of two cases can be diminished below every given magnitude in the data or in what is posited, it must also be possible to diminish it below every given magnitude in what is sought or in what results'; or to speak more familiarly: 'When the cases (or what is given) continually approach and are finally merged in each

[65] G., Vol. V, p. 460; Russell, p. 282; U., p. 266; Couturat, 1901, p. 1.

[66] G., Vol. V, p. 461; Russell, p. 283.

[67] G., Vol. III, pp. 51-55; Russell, pp. 64, 222. Cf. Cantor, Vol. III, pp. 277-278, 367; G., Vol. IV, p. 229; Couturat, 1901, pp. 233-237; Latta, pp. 37-39, 71, 83-84, 376-377.

other, the consequences or events (or what is sought) must do so too.' Which depends again on a still more general principle, namely: 'When the data form a series, so do the consequences (*datis ordinatis etiam quaesita sunt ordinata*).'"

Later on Leibniz also expressed his "law of continuity" by saying that "nature never makes leaps,"[68] and it would certainly appear that each of the above forms of the law implies the other. We first find an exact treatment of the question with Bolzano, and this will be mentioned presently.

Couturat[69] remarked on the first form that the enunciation was quite mathematical and that the principle was evidently suggested to Leibniz by his work on the infinitesimal calculus, "of which the first postulate is that we have to do with functions that are *continuous* and have derivatives." However this may be, it is a fact that the phrase "a function is subject to the law of continuity" used to mean throughout the eighteenth century and the first few years of the nineteenth, that the function in question was not one of those which Euler maintained could appear in the integrals of partial differential equations and which are expressed by differential equations in different intervals.[70]

For the moment I will distinguish with Arbogast between the "contiguity" and "continuity" of a function—the word "continuous" being used in the sense of Euler and the word "contiguous" in the sense in which we now, after Bolzano and Cauchy, use the word "continuous," and which seems to be the sense in which Leibniz used the phrase "varying according to the law of continuity." The

[68] G., Vol. V, p. 49. Cf. the passages quoted in Russell, pp. 222-223, and the first of the grounds against extended atoms mentioned on p. 234. Cf also *ibid.*, pp. 63-66.

[69] Couturat, 1901, p. 235 note.

[70] Cf., for example, Jourdain, in *Isis*, Vol. I, 1914, pp. 669-700.

fact then seems to be that Leibniz and his immediate successors thought that every function which could appear in analysis, geometry, or mathematical physics, was continuous and therefore contiguous; Euler made it probable that some important functions were not continuous and some of these were contiguous and some not. Fourier showed convincingly that those functions which seemed discontinuous to Euler were really continuous, since they could be represented by trigonometrical series, and thus that discontinuity was no mar to continuity. Finally, in 1814, Cauchy freed the language of analysis from the difficulty that one and the same function could be both discontinuous and continuous according to the way in which it was represented, by ignoring the notion of continuity and keeping only that of contiguity. Cauchy, in 1814, spoke of contiguity as "continuity,"[71] and this will seem to us confusing only if we do not reflect that the name "continuous" could be used by another conception as its original bearer was deceased.

It is, by the way, somewhat remarkable that Fourier should, in spite of this discovery, have clung to Euler's idea of "continuity" of a function and should have left to Cauchy the formulation of that useful property of certain functions which we still, like Cauchy, call "continuity"; but such is the fact. Especially at the beginning of his career, Cauchy was greatly influenced by the work of Fourier, and we may describe a great part of Cauchy's work by saying that it was the precise description and introduction into pure mathematics of many of the new ideas to which Fourier was led. Though we see the germs of a new conception of the "continuity" of a function in a paper by Cauchy of 1814, the conception was precisely defined by him only in 1821, and it is to Bernard Bolzano—who seems to have been uninfluenced by Fourier and very much in-

[71] *Ibid.*, pp. 688, 689, 690.

fluenced by Leibniz—that the priority of a precise formulation of the new conception of the "continuity" of a function must be attributed.

In a paper published in 1817,[72] Bolzano criticized the statement that, because a function "varies according to the law of continuity," it must pass through all intermediate values before it can attain to a higher one, on two grounds. In the first place, this is a provable theorem,—if, as he seems tacitly to imply, the following "correct" definition of "continuity" is used. In the second place, in the above statement "an incorrect conception of *continuity* is taken as basis. According to a *correct explanation* of the conception of continuity, we understand by the phrase: 'a function $f(x)$ varies according to the law of continuity for all values of x which lie inside or outside certain limits,' only that, if x is any such value, the difference $f(x-\omega)-f(x)$ can be made smaller than any given magnitude if ω may be taken as small as we wish."

XI.

In somewhat close connection with the work of Leibniz on mathematical logic stands the work of Johann Heinrich Lambert,[73] who sought—not very successfully—to develop the logic of relations. Toward the middle of the nineteenth century, George Boole[74] independently worked out and published his famous calculus of logic, which is almost exactly what Leibniz would have called a *calculus ratiocinator*. At the same time as Boole, and independently of him or of anybody else, Augustus De Morgan began to work out logic as a calculus, and later on, taking as his guide the maxim that logic should not consider merely certain kinds of deduction but deduction quite generally,

[72] See the further account and references, *ibid.*, pp. 695-697.

[73] See the historical parts of John Venn's *Symbolic Logic*, London, 1881; 2d ed., 1894, quoted by Jourdain, *Quart. Journ. of Math.*, Vol. XLI, p. 332.

[74] Cf. Jourdain, *loc. cit.*, pp. 332-352.

founded all the essential parts of the logic of relations. William Stanley Jevons[75] criticized and popularized Boole's work; and Charles S. Peirce, Richard Dedekind,[76] Ernst Schröder, Hermann and Robert Grassmann, Hugh MacColl,[77] John Venn, and many others, either developed the work of Boole and De Morgan or built up systems of calculative logic in modes which were largely independent of the work of others.

But it was in the work of Gottlob Frege, Guiseppe Peano, Bertrand Russell, and Alfred North Whitehead, that we find a closer approach to the *lingua characteristica* dreamed of by Leibniz. To this work other articles in this number will be devoted.

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[75] Cf. Jourdain, *loc. cit.*, Vol. XLIV, pp. 113-128.
[76] Cf. *Monist* for July, 1916, pp. 415-427.
[77] Cf. Jourdain, *loc. cit.*, Vol. XLIII, pp. 219-236.

although Leibniz makes more clearly than Descartes the distinction between *vérités éternelles* (*a priori*) and *vérités de fait* (*a posteriori*).

In connection with this method and with the mechanism of Descartes, we can observe Leibniz's dependence in his estimate of mathematical symbolism. His scheme for a universal philosophical language appears to have been made out before he saw Descartes's letter on the subject. But there is no doubt of the source of the high value given by Leibniz to mathematics as a guide to philosophical method. It is in part the common thought of the age which had achieved so much by the application of exact mathematical reasoning to the data of physical science. The nature of things seemed to be disclosed when the master-key of calculation was used. "I believed," says Descartes, "that I could borrow all that was best both in geometrical analysis and in algebra and correct all the defects of one by the help of the other." Leibniz carries this conception further by arguing that we could make for philosophy a real symbolism (*caractéristique*), like the numbers in arithmetic or the signs in algebra. "If we had a symbolism," he says, "we should be able to reason in metaphysics and morals in much the same way as in geometry and analysis." And as mathematics has developed because of the signs we have invented, so philosophy would grow by the adoption of symbolic logic. What we need, he says, is not a vague statement concerning the limitations of reasoning, but an exact method. All reasoning is calculation, but, as against Descartes, it is not therefore mathematical. As far as men do think effectively in philosophy their thinking is "mechanical"; it is the primitive nature of the mechanism which is the source of the trouble. How happy would philosophers be if they adopted the universal symbolism; for then "when a dispute arose, it would suffice to take their pencils in their hands, to sit down to their slates

and to say to each other, with a friend as witness if they liked: Let us calculate."[2] This situation Leibniz imagined himself to have all but reached. "In the general characteristic or universal calculus," he says, "I have definitions, axioms and very remarkable theorems and problems in regard to coincidence, identity, similitude, relation, power or cause, and substance, and everywhere I advance with symbols in as precise and strict a manner as in algebra."[3] This is in the mood of the Cartesians who hoped to explain everything *more geometrico.*

In the second place, Leibniz's psychology is closely related to that of Descartes. We may omit the discussion of the epistemological criticism of Descartes's *Cogito, ergo sum.* It belongs to the general body of Leibniz's positive philosophy, and is important as connecting Descartes with Leibniz only in so far as Leibniz says that one's own existence is *not* a premise for necessary truths and is not any more certain than the existence of one's thoughts.[4] In psychological description Leibniz emphasizes the fact of unconscious perception, accepting more or less exactly the Cartesian idea of perception. Thus "perception" should be distinguished from apperception or consciousness. "In this matter the Cartesians have fallen into a serious error, in that they treat as non-existent those perceptions of which we are not conscious." Unconscious mental states are therefore added to the list of psychological facts; and their existence is used to show the nature of some monads. But in the main the Cartesian, as opposed to the scholastic, psychologically is accepted.

Thirdly, as to the use of mechanical conceptions for physical science, this was of course not peculiar to Descartes and Leibniz. It was the common ground of all

[2] G., Vol. VII, p. 200, quoted in Russell.
[3] Letter to Arnauld, Jan. 14, 1688. Cf. Montgomery, p. 241.
[4] Russell, § 102.

who made progress in the understanding of nature in the seventeenth century. Against the vagueness of the metaphysical physics inherited from the Middle Ages, it was effective not only upon grounds of general reasoning but also in the results it had to show. Leibniz, more even than Descartes, valued such results and in that he followed the ideals of Bacon but, naturally, with more intelligence. He felt that the bearing of scientific investigation upon the ordinary task of human life was not unimportant. The debt Leibniz owed to Descartes is acknowledged to have been great even before Leibniz came to Paris; but in Paris he seems to have taken up the new scientific and mathematical method with renewed energy. In the letters to Malebranche[5] he puts his position most clearly, and in one of 1679, which was apparently never sent, he writes: "It seems that all the harvest of Descartes's philosophy is over, or that the hope that was in it has perished in the bud with the death of its author; for the majority of Cartesians are only commentators." In the same letter he says of Descartes, "there are perhaps few who perceive as clearly as I the greatness of his mind," but "his geometry is what I think least valuable in Descartes." And in the letters to Arnauld he is continually correcting or criticizing the geometry of Descartes. The hope that there might be great practical results had been frustrated, and even the theoretical development seemed lacking. But there is no hesitation in Leibniz as to the value of the geometrical conceptions of science. It is true that these seemed to imply a complete removal of the "spiritual" and the "supernatural" from the regions dealt with in science, and old final causes would also disappear. Even this, however, although it was a reason for Leibniz's ultimate repudiation of Cartesian metaphysics, could not shake his belief in Cartesian physics.

[5] G., Vol. I, pp. 334f.

In the purely metaphysical issue Leibniz seems to grant to Descartes the arrangement of the machine of the universe by God; but he makes a small change, for "it is more reasonable and more worthy of God to suppose that he has created the machinery of the world in such a fashion from the very start that without doing violence at every moment to the two great laws of nature, that of force and that of direction, but rather by following them exactly (except in the case of miracles), it so comes about that the internal springs of bodies are ready to act of themselves, as they should, at the very moment when the soul has a conforming desire or thought."[6] The whole conception here is Cartesian—the machine and the springs of bodies and the desiring soul. The suggestion that what is "worthy of God" is true may perhaps be regarded as Leibnizian; but even that is in some part shared by Descartes, in succession to the established tradition according to which Aquinas long before could prove that Paradise was in the east, because the east is more "noble." At this point, however, Leibniz parts from Descartes and endeavors, still with an eye to Cartesian influences, to render experience not as mechanism with a parallel mentalism, but as activity and pre-established harmony. He was, in his own conception, restoring teleology to metaphysics and spirit to nature. But the result must be dealt with in the discussion of Leibniz's system as a whole. He follows Descartes at least so far as to begin with his description of facts.

The debt of Leibniz to Descartes is perhaps not less great in that metaphysical issue upon which they differ fundamentally—the conception of substance. It seems to be possible that the true source of concepts each had is not yet fully investigated by historians; but as the problem is now generally stated, Descartes stands for (*a*) extension as the nature of one kind of substance; and also, by implica-

[6] Letter to Arnauld, April 30, 1687.

tion, for (*b*) the real unity of all separate "things" in one (as in Spinoza's theory) or in two forms. Against this Leibniz stands for (*a*) activity or *actus*, in the scholastic sense, as the ultimate nature of all existents, and (*b*) individual units called monads, which are not in any sense less real than the whole within which they are related. The opposition does not involve a complete reversal of views, although Leibniz writes, "Extension is nothing but a certain indefinite repetition of things in so far as they are similar to each other and indiscernible. It presupposes things which are repeated." In such words he seems to imply that he had "reduced" extension to monads. Probably the source of both Descartes's and Leibniz's reasoning on matter is to be found in the theories of late scholasticism, sometimes called nominalism.

At this point we may perhaps note the relation of Leibniz to scholasticism, for it is not very different from that of Descartes, who had made less extensive study of its literature but was not for that reason any less affected by its leading conceptions. It is usual to consider scholasticism chiefly as a system of logic. The Aristotelian syllogism and the philosophical method, in part misleading, with which the name of Aristotle was connected, did undoubtedly color the whole of the medieval tradition in philosophy and science. This had been brought to an almost absurd elaboration by Lullus: and all this Leibniz acknowledges to have greatly impressed him. But this is not the most valuable part of scholasticism, nor is it the point in which scholasticism has most importance in the history of philosophy. For, first, we must recognize that scholasticism did not mean in the seventeenth century the theory of Aquinas and Scotus only, or chiefly, but the theory of Ockham. There had been a revival of Thomism, but in the main the philosophical tradition was such as Ockham had left it, and in the *matter* as opposed to the *method* the

new thought of the Renaissance depended upon what the histories of philosophy usually call nominalism or conceptualism.

The idea of *extension* as the nature of "substance" is to be found in Ockham. Thus "quantity" (used as meaning extension) is not distinct from "substance."[7] The point cannot be argued here; but in Ockham's eagerness to be rid of "quantity" as a real thing, he seems to have persuaded himself that the reality which most people call substance was quantity or extension. The influence on Descartes may have been very indirect or even unconscious. On the other hand Leibniz is generally recognized to have owed the conception of *actus* or activity to the scholastics. In his letters to the Jesuit Des Bosses this is abundantly clear. It is not, however, sufficiently noticed that this activity is conceived as individualized also because of the late scholastic tradition, for which again the name of Ockham may be taken to stand. The word "monad" may have been due to Giordano Bruno; but the phrase should not be forgotten by which Ockham revolutionized scholastic metaphysics: "Everything outside the mind is in itself individual in such a way that *itself* without any addition (e. g., the principle of individuation, etc.) is a 'this.' "[8] Leibniz's monadism is at least in part affected by this suggestion; but he does indeed often go back to the Thomistic influence when he "explains" individuation of the finite monad by some process of connection with "matter" or *potentia*. He might have maintained with the late scholasticism that the individual (*illud quod*) is the only substance and needs no further "explanation." But whatever the source of his thought, Leibniz clearly allowed for ultimately real individuals and he granted the calculability of phenomena in terms which imply all that Descartes intended to indicate by "extension." The rest of his doctrine was not Cartesian;

[7] *De sacramento altaris*, q. 3. [8] *In sententias*, q. VI.

but his continual attacks upon the idea that extension is the ultimate nature of matter should not blind us to the amount of general agreement between Descartes and himself, at least as opposed to the official and established conceptions of the day.

The relation between the two may perhaps be put in this way: For Descartes the calculability of phenomena is fundamental and there is nothing more to be said about it. The *nexus* between things is mechanical, in the sense that origination or spontaneity within the system may be left out of account in our description of the material universe. And this position has become so familiar to us that it is hardly valued by philosophers except for exercising their wits in discovering in what sense it may be mistaken. This position was known to Leibniz as the new doctrine which had overcome the "spiritualism" of the medieval metaphysics of nature. And undoubtedly he saw its complete validity for the description of phenomena, or the explanation of them in so far as that can be had by showing how they are connected. He disputed the details of the Cartesian geometry, but he granted the calculability of phenomena.

Descartes had, however, left upon his hands, so to speak, the kind of substance which was called soul or mind, thus creating a problem as to the connection of soul and body solved in one way by the occasionalists and in another by Spinoza. Leibniz attempts to avoid the difficulty by beginning with one type of ultimate reality, active substance. The meaning of this can only be rendered in a full account of Leibniz's philosophy. But even in Leibniz's monadism appear the automata and machines of Descartes. Thus he says,[9] "Every organic body of a living being is a kind of divine machine or natural automaton;" and again,[10] "Descartes saw that souls cannot impart force

[9] *Monadology*, 69. [10] Ibid., 80.

to bodies because there is always the same quantity of force in matter. Yet he thought that the soul could change the direction of bodies. This was, however, because at that time the law of nature which affirms also the conservation of the same total direction in the motion of matter, was not known. If he had known that law he would have come upon my system of preestablished harmony. According to this system bodies act as if (to suppose the impossible) there were not souls at all and souls act as if there were no bodies, and yet both body and soul act as if the one were influencing the other." There is much force in your "as if"! But in any case Leibniz grants that one may neglect soul in describing bodily changes, the correctness or incorrectness of which it is not our present purpose to discover. The important fact for us here is that it was only by retaining the Carestian doctrine as to natural phenomena being in some sense (whether fundamentally or superficially) calculable, that Leibniz was able to contribute to the progress of our knowledge of the universe. It is beside the point, in this regard, to ask what metaphysical truth is implied in the processes of physical science. That problem may be solved or left unsolved while the undeniable fact must be recognized that the Cartesian hypothesis has led to a control of natural forces and a power of prediction which can hardly be refused the name of knowledge.

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THE DEVELOPMENT OF LEIBNIZ'S MONADISM.

THE study of the *Monadology* may be comprised in three stages. In the first we isolate the work; with no other aid than the philosophical counters which itself employs, we attempt to draw its fantastic world around us and find it real. Perhaps we supplement it by searching in other works of Leibniz for elucidations of points which are not clear; but in any case we take the *Monadology* as a creed and test our possibilities of belief. No philosophy can be understood without this preliminary effort to accept it on its own terms; but its true value can never be extracted solely in that way. The perfected or the summarized form of any system is the starting point, not the terminus of study. We must effect a radical restatement, find in it motives and problems which are ours, giving it the dignity of a place in the history of science when we withdraw from it the sanctity of a religion. In losing the consistency of a closed system, it gains the consistency of reason, is attached to something larger than itself. Russell and Couturat have accomplished this revaluation for Leibniz. But beside the leading motive, the reason of a philosophy, there are other strata both below and above: prejudices, traditions, suggestions, motives which imperfectly assimilate to the central motive, all of which combine to give to the system the form which it has. The present essay is merely a preface to the investigation of these forces.

There are influences of suggestion, influences of tradition, personal influences, and, moreover, there is more than one conscious interest. Among influences of the first sort upon Leibniz (none of them of the highest importance)

I should class a variety of authors whose contributions to Leibniz are more verbal than profound. Leibniz's reading was wide beyond any point of selection, and he appears to have derived some entertainment from such philosophers as Giordano Bruno, Maimonides, and the Averrhoists.[1] Bruno is a classic example of influence in the most superficial sense. It is not certain, nor is it important, at what period Leibniz became acquainted with Bruno's works. For the probability that Leibniz was struck by the figurative language, that Bruno may have been in the background when Leibniz wrote some of his more imaginative passages, there is evidence enough. For the probability that Bruno affected Leibniz's thought there is no evidence whatever. What we have is a statement which bears strong superficial resemblances to the statement of Leibniz; the arguments, such as they are, the steps which lead up to the statement, are not similar. Leibniz's arguments are sufficiently strong not to demand support from the fact that there were monadologists before Leibniz. To his imagination we may concede plagiarism. But it is with the sources of his thought, not with the sources of his imagery, that we are concerned.

The other sources mentioned may be dismissed in the same way. It is interesting, perhaps, but not valuable, to observe that Leibniz read with appreciation a book by Maimonides. And though he never couples the names of Spinoza and Maimonides together, the notes which he made upon this book single out just the points of resemblance to the *Theologico-politicus*—the first work of Spinoza that he read. He was interested in Hebrew and Arabic studies. Bossuet sends to him for a translation of the Talmud. He announces to Bossuet a translation of the Koran. A dialogue of 1676 shows that he knew,

[1] For Bruno see H. Brunnhofer, *G. Brunos Lehre vom Kleinsten.* For Maimonides see Foucher de Careil: *Leibniz et la philosophie juive;* Rubin: *Erkenntnistheorie Maimons.*

through Maimonides, the doctrines of the Averrhoists and of a certain Jewish sect, the Motekallem. In 1687, while traveling in Bavaria, he undertook some study of the Kabbala, and perhaps noticed the theory of emanation from an infinite being which consists in an indivisible point—and the microcosm is said to be a familiar idea in Jewish philosophy. These studies, rather shallow it is true, illustrate Leibniz's insatiable curiosity toward every sort of theological hocus-pocus. Monadism was probably a satisfaction of this side of Leibniz's mind, as well as the outcome of his logical and metaphysical thought.

Of influences of suggestion there is only one which may have been of the first importance—the influence of Plato, to be treated later. The main influences which directed Leibniz are of three kinds: the scholastic Aristotelian tradition in which he was brought up, the very early stimulus of a personal teacher toward a mathematical conception of the universe, and Leibniz's temporary adhesion to atomism. His chief motives, more or less corresponding to this classification, were theological, logical and physical.

Merz expresses the conventional opinion[2] in saying that the *De principio individui* "bears witness to the young author's knowledge of scholastic learning as well as to his dexterity in handling their dialectic methods." Incompetent to impugn the scholastic erudition of young Leibniz, a perusal of this document impels me to exclaim with Kabitz, "as if the copious citation of passages from scholastic compendia proved any 'astonishing' learning on the part of Leibniz; as if he could not obtain these quotations just as well second-hand!"[3] The treatise is very short and very dull. Two or three passages in it are often quoted. "Pono igitur: omne individuum sua tota entitate individuatur"; and "Sed si omnis intellectus creatus tolleretur, illa

[2] Merz, p. 15.
[3] Kabitz, *Entstehung der Philosophie des jungen Leibniz*, p. 50.

relatio periret, et tamen res individuarentur, ergo tunc se ipsis." The principle of individuation is not mental, nor is it negative. Though Leibniz documents this work with such names as Occam, Scotus, Aquinas, Suarez, Molina, Zabarella, what the thesis shows is not extent of learning or originality of thought. It shows that there was a certain body of inheritance which pointed in a certain direction. It shows a scholastic point of view from which Leibniz never really escaped, and which he never wholly rejected.[4] In the light of these quotations is to be interpreted not only monadism, but the materialistic atomism which for a time engaged his attention. At this early period, and indeed throughout his life, there is little evidence of direct adaptations from Aristotle. But here as always one finds the acceptance of the problem of substance, transmitted from Aristotle through the form which the school had given it. In some ways diametrically in opposition with Aristotle, this scholastic view of substance which Leibniz held is yet an Aristotelian inheritance. This point is of capital importance.

It appears that Leibniz abandoned his study of the philosophers of the church when he felt called, at a very early age, to the mechanical view of nature (Merz, p. 15). But there was never a complete renunciation, and Leibniz, who seldom spoke ill of a dead philosopher, always praises the schoolmen. The change was a transition and not an apostasy. In 1663, at Jena, while pursuing his studies in jurisprudence, he fell under the influence of Weigel. Weigel was acquainted with the work of Copernicus, Kepler and Galileo. Kabitz says (*op. cit.*, p. 112) that "the fundamental conception of Leibniz's system (according to which the universe is an harmonic, mathematico-logical related whole .... became a firm conviction with Leibniz

[4] Nolen, *Quid L. Aristoteli debuerit*, p. 27: "mea doctrina de substantia composita videtur esse ipsa doctrina scholae peripateticae. Nisi quod ille monadas non agnovit."

through Weigel, before he was acquainted with the work of Hobbes." Bisterfeld of Leyden is another mathematician admired by Leibniz in his youth, and his influence is supposed to be visible in the *Ars combinatoria*. The idea of a harmony of a universe of individual substances is present in other writings of Leibniz's adolescence.

Leibniz's scholastic training in metaphysics under Thomasius was followed by that period in which, as he says, "having freed myself from the yoke of Aristotle [by which he means the attenuated scholasticism of his day], I took to the void and the atoms, for that is the view which best satisfies the imagination."[5] This may have been about 1666.[6] It is easy to see from the *De principio individui* (written, according to his own chronology, when he had already fallen under the influence of Gassendi) that this liberation was merely a development of extreme nominalism in the currents of his time. In 1676 he can still write, "Ego magis magisque persuasus sum de corporibus insecabilibus. . . . simplicissima esse debent ac proinde sphaerica," but goes on to say "Nullus enim locus est tam parvus quin fingi possit esse in eo sphaeram ipso minorem. Ponamus hoc ita esse, nullus erit locus assignabilis vacuus. Et tamen Mundus erit plenus, unde intelligitur quantitatem inassignabilem esse aliquid."[7] The atomism survives in 1676, although the void is abandoned, and the influence of his mathematical work is visible (this was just at the end of the period in Paris, when he was corresponding with Newton through the medium of Oldenburg). In this year occurred also his visit to London and to the Hague.

In the next period of his life, when he had for some years been occupied chiefly with mathematical matters, falls the elaboration of his argument against Descartes's theory of matter,—Descartes, who had been partly responsible for

[5] Latta, p. 300. [6] See Kabitz, p. 53. [7] Couturat, 1903, p. 10.

Leibniz's tendency toward a mechanical view. The unsatisfactory character of the views of Descartes and of Gassendi had, it is true, been pointed out by him several years before. In this later period, besides physic and pure mathematics, a third scientific interest may be noted. He refers often to Swammerdam, Leuwenhoek and Malpighi, and it is evident that he felt a genuine enthusiasm for the progress of biology, aside from the support which certain theories lent to his doctrine of preformation. But as his interest in biology is apparently subsequent to the observable beginnings of monadism, these theories were rather a confirmation than a stimulus.

To these philosophical and scientific occupations must be joined another which was no less important. This is his perfectly genuine passion for theology. Developed perhaps out of his early training, this theology, in a mind which never lost an interest it had once taken up, remained a powerful influence throughout his life. His solicitude for the orthodoxy of his philosophy was not merely policy or timidity; his theological disputations are not merely a cover for logical problems. Leibniz's theological motive is responsible for much of the psychology of his monads; it took deep root in his system, though not altogether without disturbance of the soil. The only two interpretations of Leibniz which are of any importance, that of Dillmann[8] and the superior interpretation of Russell and Couturat, minimize the significance of this motive.

"Ma métaphysique est toute mathématique, pour dire ainsi, ou la pourroit devenir," Leibniz writes to the Marquis de l'Hôpital (Dec. 27, 1694). And Russell says (p. 49) in speaking of the subject-object relation, "the whole doctrine depends, throughout, upon this purely logical tenet." Strictly speaking, this assertion is perfectly justified. For a historical account it is insufficient. Leibniz puts his prob-

[8] *Neue Darstellung der Leibnizischen Monadenlehre.*

lems into logical form, and often converts them slyly into logical problems, but his prejudices are not always prejudices of logic. The value of Leibniz's logic is to a certain extent separable from the value of his philosophy. The view of the nature of substance with which he starts is due to a logical problem. But there is no logical descent from pluralism to the view that the ego is substance. Leibniz's view of substance is derived from Aristotle, but his *theory* of substance is quite different: it is Aristotle's theory filtered through scholasticism and tinctured by atomism and theology.

When we father the problem of substance upon Aristotle, we must remember that it was a problem which he never succeeded in resolving, or pretended to have resolved. The chief inheritance of modern philosophy from his doctrine is the proposition that "substance is that which is not predicated of a subject, but of which all else is predicated" (1029*a*). Aristotle recognizes that there are various senses in which we may use the term, and various substances beside the sensible substances, which have matter. In one sense the composite of form and matter (e. g., animals and plants) is substance, in another sense substance is "the form by which the matter is some definite thing" (1041*b*). And again the substratum (1029*a*) is that of which everything is predicated. Matter certainly is not substance, because matter *qua* matter has neither limit nor the potency of limit by separation (see 1017*b*). And again the universal is more substantial than the particulars (*Metaph.*, Z 13). Wherever Aristotle pursues the concept of substance it eludes him. These tentative definitions, assumed for dialectic purposes, are abandoned in favor of that of 1041*b*. This bears, it is true, very striking resemblances to the substance of Leibniz. As to the meaning of form and the relation of formal to efficient and final

cause Aristotle remains difficult and vague, while for Leibniz the formal and efficient causes in the case of substance are identical.

There is another and very serious difficulty in the theory of Aristotle. From one of Aristotle's points of view only the individual should be real, from the other only the specific. The form is always ἀτομόν; thought analyzes and resynthesizes its constituents to give the λόγος τοῦ τὶ ἦν εἶναι. Of the subject either the whole or a part of the definition can be affirmed: thus we can define Socrates *qua* man as ζῷον δίπουν λογικόν. But predications of particular individuals belong to the attributive, not to the definitory type of judgment. In this type of judgment the predicate affirmed, although it belongs to the subject, is not a constituent of the subject's essential nature. As the essential nature of Socrates is man, anything which is not contained in the form of man in general will be attributive only and not definitory, inasmuch as it might have been otherwise. For Aristotle not all predicates are contained in their subjects. Hence there can be no definition of individuals of a species (1040*a*). The substance must be individual, in order to be the subject; it must be a "this." But the "this" cannot be composed of universals, because no number of "suches" will constitute a "this," and on the other hand it cannot be composed of other substances. We thus get two opposed views: the substance is the form of the species, in which case it breaks loose from the concrete thing and gives rise to the same difficulties which Aristotle censured in Plato; or the substance is the individual thing, in which case there is no definition and no knowledge. One view is in harmony with Aristotle's methodology, the other with his theory of elementary cognition.[9]

[9] In *An. post.* 100*a* (Chap. XIX) we are told how the knowledge of the universals arises through experience of particulars. "First principles" are arrived at by induction. What is not made clear is the status of the particulars after scientific knowledge is established.

Aristotle is here betrayed by his representation theory —the exact correspondence between constituents of propositions and constituents of things; although in other contexts he is an epistemological monist. The same incoherence appears in his account of the soul. Is the substance the compound of matter and form, or the form alone?

It was the Aristotelian problem of substance, affected by scholasticism, that Leibniz took upon his shoulders at the beginning of his career. Later in life he observes that he has been re-reading Aristotle, and that he finds much of value in him. The extent of his acquaintance with the text may be left in doubt. It is probable that he had little or no direct knowledge, that he abandoned the study of the history of philosophy almost altogether for some years, and the fresh approach to Aristotle did not produce much effect upon his subsequent work. The interest lies in Leibniz's saturation which the Aristotelian tradition—in spite of a momentary peevishness against the degenerate scholasticism in which he had been brought up—and in the compound to which the contact of this training with the speculations of contemporary science gave rise. To this particular problem the drawing of parallels and the estimating of borrowings—conscious and unconscious—is irrelevant. Nor are we here concerned with the question whether "this seemingly fantastic system can be deduced from a few simple premises."[10] The question is the actual genesis of the system. If, at the age of fifteen, Leibniz inclined to the view that substances are particular individuals and that relations exist only in the mind; if we can see that his transition to atomic materialism follows quite easily from this; if we find that his further development depended upon the way in which his scientific researches and his theological prejudices — largely an inheritance from his early training—played into each other; then we shall con-

[10] Russell, p. viii.

clude that his metaphysics and his scientific achievements —logical and mathematical—are two different values.

What is curious about Leibniz's mind is the existence of two distinct currents. As a scientist he has a clear and consistent development. Every step is justified and coherent from this point of view alone. His metaphysics is carefully built upon his scientific evolution. On the other side is a strong devotion to theology. His study of Descartes marks a stage in the development of both. Descartes's theory of matter, and Descartes's theory of self-consciousness both had their effect upon him. And it is always the same mind working, clear and cold, the mind of a doctor of the church. He is nearer to the Middle Ages, nearer to Greece, and yet nearer to us, than are men like Fichte and Hegel.

We have seen that there is a very great difference between the Aristotelian theory of substance and the nominalism deriving from it with which Leibniz starts. Both in the *Metaphysics* and in the *De anima,* it is true, Aristotle leaves the answer somewhat ambiguous. When he discusses the substance of organic beings we are apt to think that each individual is a substance—that the form of each body is an individual—one form for Socrates, and another for Callias. It is difficult to avoid this conclusion, but in general, for Aristotle as well as for Plato, whatever was merely individual was perishable and incapable of being a subject of knowledge. But if we say, with Burnet (*Greek Phil.*, p. 331) that "Plato found reality, whether intelligible or sensible, in the combination of matter and form and not in either separately," and take the same view of Aristotle, yet we cannot say that they found it in each individual as a world apart. This is an instance of the differences between Leibniz and the Greeks. In Leibniz we find the genesis of a psychological point of view; ideas tend to become particular mental facts, attributes of par-

ticular substances. If the form or principle of Aristotle were different in each man, this form would be Leibniz's soul. For the Greek the human was the typically human, individual differences were not of scientific interest; for the modern philosopher individual differences were of absorbing importance.

We may now trace the two currents which are imperfectly united in the monad. Leibniz approaches the problem of substance primarily as a physicist. "Leibniz does not begin with the problem, what is the substance of the body, what is its origin, but from this: how the principle of the body itself may be conceived" (Dillmann, p. 63). To those readers—there are still a few—who know Leibniz only through the *Monadology,* the steps to the conclusion will remain unknown. Unless we appreciate the original question we shall be unable to understand his solution of the problem of body and soul, and of the problem of our knowledge of external objects. He never asked the question, "do physical bodies exist?" but always, "what is the principle which makes physical bodies intelligible?" The answer is found in his reaction to Cartesianism. And at this point, while the problem of energy was engaging his attention, he read some of the dialogues of Plato, and was confirmed in his conclusions especially by certain parts of the "Sophist." What we get is on the one hand an explanation of the principle of matter, and on the other an idealistic metaphysic, largely influenced by Descartes, based upon self-consciousness. The latter aspect has of course been more exploited than the former.

Leibniz's account of physical matter is a much more scientific, but in some respects much cruder, explanation than Aristotle's. For Aristotle's account is fundamentally a relativistic one, i. e., "matter" has various meanings in relation to shifting points of view which form a series but are not themselves defined. There are meanings in various

contexts, but no absolute meaning; and the series of points of view, the series of contexts, has no absolute meaning either. One misses the whole point of Aristotle's theory if one regards matter as a "thing." It is—whether as primitive matter, as the four elements, or as any compounds (I mean σύνθεσεις not μίξεις) of any degree of complexity formed out of these, one side of a contrast in the mind (or imposed upon the mind) though this mind is no more absolutely definable than matter itself. (Hence Aristotle is neither an idealist, in the modern sense, nor a pragmatist.) *Materia prima* is not simply negative nor is it positive in any apprehensible way. It is simply the furthest possible extension of meaning of a concept which has arisen out of practical complexes. The next stage in the conception of matter, it will be recollected, is that of a subject possessing two out of two pairs of opposites (wet-dry, hot-cold). The *materia prima* is not *actual*, because it has no predicates; the smallest number of predicates which an actual existent can have is two. That is, whatever is merely hot, or merely dry, is not a substance but is identical with the quality itself; but whatever is hot and wet, or cold and dry, is a substance different from its predicates. These elements—the possible combinations of four qualities—are capable of transmutation into one another in a cycle which occurs in the exchange of qualities (the hot-dry becomes hot-wet, the hot-wet becomes wet-cold, etc.). The third stage of matter is that of the stable compounds of the four elements held together in various proportions. This progress is not a chemical theory in the modern sense; it is a series of points of view. The formal cause is therefore identical with the thing itself, and whether the form is there is a question of what we regard as the thing. The lump of marble is a σωρός of higher compounds of the four elements—or it is a statue. One must keep in mind the two apparently inconsistent

propositions: (1) there are no forms of individuals,[11] (2) the form and the matter compose one whole.

Aristotle is too keen a metaphysician to start from a naive view of matter or from a one-sided spiritualism. To a certain extent Leibniz keeps this middle ground too. But his metaphysics tends to fall apart, as the result of his inherited nominalism, and the fissure between his scientific and his theological interests. Starting as a physicist, Leibniz naturally assumes that matter is not a relative term but that it is (if it exists at all, of which he has no doubt) something absolute. The substantiality of matter consists then (after his defection from Cartesianism) in the concept of force. Force is not conceived as something behind matter, which could be actual without matter. But neither is it a "form" in quite the Aristotelian sense. The "real and animated point" of the *Système nouveau* is from an Aristotelian point of view merely another individual, or a form of an individual. It is purely and simply a physical explanation. It involves no theory of knowledge, because it does not take into account the point of view of an observer; it is a contrast not between matter and form, but between a particular substance and its states.

The distinction between *materia prima* and *materia secunda* (of bodies) is superficially Aristotelian. But it is really only a distinction between two ways in which matter may be considered for the purposes of the physicist. It is a distinction of uses and not of contexts. "Matter is not a relative term. The ancient distinction between matter and form does not correspond to the modern distinction, since Descartes, of matter and spirit. And the dichotomy is as strongly marked in Leibniz as in Descartes. His solution of the difficulty marks the wide gulf that separates modern from ancient philosophy. For Aris-

[11] Except of course eternal and unique individuals, like the moon, which is the only individual of its species. And for later theology, the angels.

totle matter and form were always relative, but never identical. For Leibniz matter and spirit are absolute reals, but are really (as for Spinoza) the same thing. The difference for Leibniz is that between internal and external aspects. *Materia prima* is not a stage, it is an external aspect, and even for physics he finds this aspect insufficient. He is therefore led gradually into a metaphysical conception. But from this metaphysical account of the nature of the physical universe to his doctrine of souls there is really no legitimate inference.

The theory of forces, as the substances of which material changes are the states, is not the theory of the soul which derives from his more theological interest. It is, as we have said, simply an analysis of the physical universe. Had Leibniz been quite consistent he would have gone on to explain organic and conscious activity on a strictly physical basis. This he did accomplish in some measure. His doctrine of expression (see letter to Arnauld, Oct. 6, 1687) is an account of perception consistent with a purely physical and mathematical point of view. But his transmigration[12] of human souls is muddled by the identification of soul, in the sense of personality, with the animated point; of the core of feeling of the self with the force of which it is predicated. From his physical point of view he cannot arrive at self-consciousness, so that his doctrine of force has two grounds—the theory of dynamics and the *feeling* of activity. If we refuse to consider self-consciousness a simple and single act, if making an object of oneself merely means the detachment and observation of particular states by other states, then the "force" slips out of our hands altogether. It remains "internal," it is true, in contrast with primary matter, but its internality is not a character of self-consciousness. And in this

[12] Leibniz of course explicitly repudiates any "transmigration" of monads. But when he comes to the human soul its adventures seem to be tantamount to this.

event the whole theory becomes completely naturalistic. Something is the subject, but it is not the *I* which I know, or which anybody knows. And there then remains no reason why we should longer maintain a plurality of subjects. Force becomes one. Against such a conclusion Leibniz was set, (1) because it ceases to have any value for physics, and (2) because it interferes with our claim for personal immortality. Theology and physics join forces (so to speak) to rob metaphysics of its due.

Hence two curious difficulties arise. An animated force, a monad, tends to become an animated atom. The monad exerts its activity at a point in space and time. Artefacts, as for Aristotle, are merely groups of monads without a dominant monad. Organic bodies are groups with a dominant monad. In the latter case, in the case of a human being, in what sense is my body *mine,* since it is also the bodies of other monads? The dominant monad should be the form of the body, instead of which it bears a strong resemblance to a larger or more powerful cell, and the soul would have to be located, like Descartes's, in a particular place. Russell, in contrasting Leibniz's two conflicting theories (pp. 149-150) says of the second view: "in the other theory, mind and body together make one substance, making a true unity." So they ought to do. If the mind cannot make the body into a *unum per se,* instead of a mere aggregate, the original physical theory has advanced to a point at which mind and body fall apart. The second view appears to descend from Aristotle.[18] The first appears to descend from atomism. From neither philosophy does Leibniz ever shake himself quite free.

There is, from the physical side, a sense in which the monad is truly immortal. Force is indestructible, and will continue in various manifestations. But force in this sense

[18] Leibniz actually says (letter to Arnauld, July 14, 1686): "The soul is nevertheless the form of the body."

is entirely impersonal. We cannot conceive of its persistence except by associating it with particular particles of matter. Leibniz is led by his difficulties almost to the point of either denying the existence of matter altogether, or else setting up a sort of matter which will be something real besides monads.

The second objection is connected with the generation and destruction of life. For Aristotle some account of generation and destruction is rendered possible by his provisional distinction between efficient and formal causes. Aristotle was not embarrassed by a belief in personal immortality, and his philosophy confines itself with fair success to an examination of the actual, the present life. But Leibniz's force is indestructible in a different sense from Aristotle's form.[14] It persists in time as a particular existence. The monad which is myself must have previously existed; it must have been one of the monads composing the body of father or mother (see Russell, p. 154). This theory has the disadvantages of practically denying the independence of mind from body and of separating monadhood from selfhood. It substitutes biological behavior for conscious activity.

Commencing with an analysis of the nature of matter, Leibniz is led to the view of a universe consisting of centers of force. From this point of view the human soul is merely one of these forces, and its activity should be reducible to physical laws. Under the influence of an Aristotelian doctrine of substance, he comes to conclusions which are not at all Aristotelian, by his nominalistic assumption that substances are particulars. From a materialistic atomism he is led to a spritualistic atomism. In this he shows again an important difference between the

[14] Aristotle and Plato, I am inclined to believe, owe their success in navigating between the particular and the universal, the concrete and the abstract, largely to the fact that "forms," "species," had to the Greek mind not exactly the same meaning as for us. They were concrete without being particular.

ancient and the modern world. It is illustrated in the prejudice of Aristotle against the differences between individuals of the same species which he ascribes to the perverse and unaccountable influence of matter. To the Greek, this variety of points of view would seem a positive evil; as a theory of knowledge, it would seem a refuge of scepticism; to Leibniz and the modern world, it enhances the interest of life. And yet the view of Leibniz comes, *via* nominalism, out of Aristotle himself.

From the point of view of physics we have a consistent explanation which represents a great advance upon crude materialism. But it is difficult to retain the separate forces unless we conceive of matter as a positive principle of individuation. Not that the doctrine of activity and passivity is wholly unsatisfactory.[15] Its effect is to reduce causality to function. And but for the Aristotelian influence, it might possibly have done so. Instead of monads we might then have had atomic particulars. But Leibniz sometimes confuses the mathematico-physical and the historical points of view. It is true that the future of the monad should be theoretically predictable. But Leibniz leaves the basis of prediction uncertain. Without recourse to mysticism, the reasons why a monad should pass from the unconscious to the conscious state, why a monad composing the body of father or mother should suddenly be elected to domination over a new body of monads, remain unsolved. We have seen that the notion of soul or spirit is not to be reached by the theory of monads as an explanation of the principle of matter. If it is part of Leibniz's inheritance we may inquire just what Aristotle's view of the soul was.

Leibniz's theory of soul is, like that of Descartes, derived from scholasticism. It is very remote from that of

[15] There are implicitly two views of activity and passivity. According to one, causality is a useful way of treating natural phenomena. According to the other, there is true activity in clear perception, true passivity in confused. This illustrates the mixture of motives.

either Plato, Aristotle, or Plotinus. For the Greeks, even for Plotinus, the soul is a substance in a sense which does not include personal immortality. For Aristotle there is no continuity between the stages of soul, between vegetable, animal and human life. And the definition of monads as "points of view" is, so far as I can see, entirely modern.

For Aristotle, according to his own explicit statement, there is no "soul" in general. As the species of figure to figure in general, so are the souls of various species of animal to "soul" in general (*De anima,* 414*b* 20 ff.). In the higher grades of soul the same functions persist, but in a form altered by the nature of the whole. The organs of different species are related by analogy—as root is to plant, so mouth is to animal, but mouth is not a development of root. The *De anima* is not so much a psychological as a biological treatise. We find in the animal the τροφὴ and αὔξησις of the plant, but completely altered in the addition of a new faculty—αἴσθησις. And these faculties are not sharp dividing lines, but in the ascending scale are used more and more loosely.[16] The natural species are immutable, and the difference does not consist in addition or subtraction of faculty.

There is a suggestion, but only a suggestion, of the doctrine of Aristotle in the three classes of monads. Even the lowest class of monad (*Monadology,* 19) has appetition. The second has feeling (sentiment) which is something more than αἴσθησις and includes φαντασία and perhaps διάνοια. The soul of man only has self-consciousness, a knowledge of eternal and necessary truths,—νοῦς. It seems very probable that this scheme was suggested by Aristotle[17] but there is a profound difference. The classification of Aristotle is on the basis of biological functions.

[16] Cf. 413 b 12, 432, and 414. Motion according to 413 is not a fourth species of the soul besides *θρεπτικόν, αἰσθητικόν, διανοητικόν.*

[17] And, in passing, it seems possible that the theory of Leibniz may have supplied a hint for the romantic evolutionism of Diderot.

These are functions of the organism as a whole, a complex substance. Plants are not ζῷα, and have no appetition. Aristotle makes much of the distinction between beings which are attached to a single place and those which move about. For Leibniz the distinction is not biological, but psychological, and is everywhere a difference of degree. The lower monads, if they had clearer perceptions, would rise in the scale. It is not a limitation of the body, but a limitation of the nature of the monad itself which establishes differences. For Leibniz the series is a continuum; for Aristotle it is not. For Leibniz desire characterizes mind; for Aristotle desire is always of the complex organism; the function of mind is solely the apprehension of the eternal and necessary truths and principles.

There is another point upon which Leibniz may have drawn his inspiration from Aristotle, and that is the "common sense." "The ideas which are said to come from more than one sense, like those of space, figure, motion, rest, are rather from common sense, that is from the mind itself, for they are ideas of the pure understanding, but they are related to the external, and the senses make us perceive them" (see Russell, p. 163). Leibniz's theory appears to be a transition between Aristotle and Kant. What Aristotle says is this: "The above (i. e., color, sound, etc.) are called *propria* of the respective senses; the perceptions common to all are motion, rest, number, figure, magnitude. These are not propria of any, but are common to all" (418*a* 17ff). Whereas Leibniz stuffs these κοινὰ into the mind, Aristotle goes no farther than to say that they are perceived κατὰ συμβεβηκὸς by all the senses. There is not, as is sometimes thought, a "common sense" which apprehends them, as the eye perceives color.[18] What is interesting in the present context is the cautious empiricism of Aristotle's

[18] Zabarella, probably the greatest of all Aristotelian commentators, is very positive on this point.

theory, contrasted with the more daring but less sound speculations of Leibniz.

The question of the relation of mind to matter is handled by Leibniz differently from either Aristotle or Spinoza. I am inclined to think that it was conceived quite independently of Spinoza. Leibniz attacks Spinoza fiercely on the ground of Spinoza's naturalism, and for his disbelief in free-will and immortality.[19] He perceives, quite correctly, that Spinoza's view of the relation of mind and body leads to a materialistic epiphenominalism. "With Spinoza the reason does not possess ideas, it is an idea." He insists that the mind and the body are not the same thing, any more than the principle of action and the principle of passion are the same thing. But he inclines to believe that the difference between mind and matter is a difference of degree, that in all created monads there is materiality. (There seems to be a relation between *materia prima* of monads and *materia prima* of matter.) Now this suggests the Aristotelian relativity of matter and form; for Aristotle the higher substances are more "formed," the percentage of crude matter seems to decrease. There is no matter and no form in an absolute sense (except the form of God, who is rather a disturbing factor). But whereas for Aristotle matter exists only in contrast with form, and formed matter may be the matter for a higher form, for Leibniz matter really exists independently of spirit, but is really spirit.

Leibniz's use of the term "entelechy" is not identical with that of Aristotle. The monad is called entelechy apparently because it is complete in itself, complete in the sense of self-sufficient; while the entelechy of Aristotle is the *completion* or actuality of something. In the *De anima* the soul is called the first entelechy of body. To be strictly consistent, Aristotle should perhaps have held that soul

[19] See Foucher de Careil: *Réfutation inédite de Spinoza par Leibniz.*

is the second entelechy, since he maintains that it is only actual when it energizes; but he is merely trying to distinguish between the form and its operation.[20] Entelechy means that the body would not be a human body without the soul. It is difficult, it is true, not to think of the soul as something added to the body (as to Galatea) or else to identify soul with the (living) body. Soul is to body as cutting is to the axe: realizing itself in its actions, and not completely real when abstracted from what it does. In the light of Aristotle's elaborate critique of earlier theories of the soul, his view is seen as an attempt to get away from the abstractions of materialism or of spiritualism with which we begin. For Aristotle reality is here and now; and the true nature of mind is found in the activity which it exercises. Attempt to analyze the mind, as a thing, and it is nothing. It is an operation. Aristotle's psychology therefore starts with psycho-physics, and ascends to speculative reason. It is only then that we perceive what mind is, and in retrospect find that it was present in the simplest sensation.

The word entelechy as used by Leibniz loses the meaning which it had for Aristotle. It becomes figurative and unimportant. Leibniz appears at first less a dichotomist than either Aristotle or Descartes. In effect, the breach between mind and matter becomes far wider than in the system of Aristotle. In order that mind may persist at all times as something distinct from the body, appeal is made to the subconscious,—a parallelism even more mystifying than that of Spinoza. With Leibniz the relation of mind and matter is closer, the relation of body and soul more remote, than with Aristotle. The weakness in Leibniz's theory of body and soul may be due to two causes. On the one hand his theological bias made separation of

[20] See *De anima*, 412*a*, 27, where δυνάμει ζωὴν ἔχοντος means having "the potentiality of functioning," not "the potentiality of soul." The above distinction between form and operation was pointed out by Zabarella.

body and soul essential; and on the other hand it was necessary, from his more strictly philosophical substances, the monads should persist after the compound substances, the bodies, which are their points of view. It is required both by his theory of substance, and by his demand for a mathematical metaphysic. The causal series which is the monad should apparently have no last term.[21] Perception (in Leibniz's general statement of expression) requires that every series should be similar both to every other series and to the series of series.[22] The same theory which demands unconscious perception seems to demand also a series which shall not terminate in time. Supposing that the destruction of individual monads shall leave the total, as an infinite number, undiminished, nevertheless the monad as a substance will have to shut up shop, and we shall be left with a number of relations relating nothing. Some sort of persistence is necessary for the system, though not the personal immortality which Leibniz is interested in supporting. It is evident that with the possibility of changes of "point of view" the meaning of prediction becomes hopelessly attenuated. Every moment will see a new universe. At every moment there will be a new series of series; but continuity makes necessary a point of view from which there shall be a permanent series of series of series.

Leibniz's theory of mind and matter, of body and soul, is in some ways the subtlest that has ever been devised. Matter is an arrested moment of mind, "mind without memory."[23] By state is not meant feeling, but the monad at any instant of time.[24] In many ways it is superior to that of Aristotle. When he turns to preformation, to the

[21] See Russell: "Recent work on the philosophy of Leibniz," *Mind*, Vol. XII, N. S., No. 46, pp. 25-26.

[22] See Russell, *ibid.*, p. 25.

[23] *Theoria motus abstracti*, 1671; quoted in Latta, p. 230. Compare the Bergsonian theory of matter as consciousness "running down."

[24] Cf. "only indivisible monads and their states are absolutely real."

vinculum substantiale, to the immortality of the soul, we feel a certain repulsion; for with all the curious fables of the "Timaeus" or the "Physics" and Aristotle's history of animals, we know that Aristotle and Plato were somehow more secure, better balanced, and less superstitious than the man who was in power of intellect their equal.

There are two other points in monadism which direct attention to the Greeks. These are the theory of innate ideas and the theory of substance as force expressed in the "Sophist." So far as the question of indebtedness goes I think that the answer is clear enough. The views which Leibniz held were forced upon him by his own premises. He undoubtedly read Plato at a time when his own theory was not yet crystallized, but he cannot be said to have borrowed. He may be given full credit for having restored to life in a new form the doctrines of Plato and Aristotle. The monad is a reincarnation of the form which is the formal cause of Aristotle. But it is also more and less. The outstanding difference is that he sets out from an investigation of *physical* force, and his monads tend to become atomic centers of force, particular existences. Hence a tendency to psychologism, to maintain that ideas always find their home in particular minds, that they have a psychological as well as a logical existence. Leibniz on this side opened the way for modern idealism. To his anticipations of modern logic of a school opposed to absolute idealism it is unnecessary for me to point. No philosophy contains more various possibilities of development, no philosophy unites more various influences. That he did not always unite them successfully—that he never quite reconciled modern physics, medieval theology, and Greek substance, is not to be reproved when we consider the magnitude of his task and the magnitude of his accomplishment.

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LEIBNIZ'S "IMAGE OF CREATION."

IN the achievements of great men the trivial and curious frequently loom higher than the solid and substantial. At one time Kepler's fame centered largely around the pseudo-discovery of fanciful relations between the regular solids and planetary distances. Placing the icosahedron, dodecahedron, octahedron, tetrahedron and cube, one within the other at such distances that each solid was inscribed in the same sphere about which the next outer solid was circumscribed, he found that the radii of the spheres were roughly in the ratio of successive planetary distances. Only in an uncritical age could such very crude numerical resemblances command any attention, especially as there seemed to exist no causal relation between said radii and the distances of planets from the sun.

Nowadays we smile at Kepler's early speculation. Nevertheless it is a fair question to ask why the regular solids should be less likely to play a part in the mathematical theory of planetary motion than does that conic section which was destined later in Kepler's career to lend itself to the establishment of his permanent world-fame as an astronomer—why the ellipse rather than the "Platonic figures"? In Kepler's time the law of gravitation could not be appealed to for arbitration; it was still hidden away in the realm of the unknown. No *a priori* decision was within reach at that time. The answer could come only after painstaking measurements, combined with mathe-

matical deduction, which, in this case, confirmed one guess and exploded the other.

The famous mathematician Sylvester derived great satisfaction from lecturing in Baltimore on versification and displaying his skill in the making of rhymes. Rumor has it that he was fonder of his grotesque booklet, the *Laws of Verse*, than of any of his great mathematical discoveries.

Thus it was also with Leibniz. He was strangely partial to a discovery of very minor importance that he made relating to the so-called binary numbers, which are constructed on the scale of 2 instead of 10 and require only two symbols, namely 0 and 1. In his scale, 1 is written 1, 2 is written 10, 3 is written 11, 4 is written 100, 5 is written 101, and so on.

The charm of Kepler's regular-solid-theory of planetary distances lay in the unexpected relation thought to exist between magnitudes so foreign to each other that not the remotest cause for such intimate relation could be imagined. Sylvester's fantastic performances lay in the acrobatic groupings of words similar in their terminal sounds. The fascination in the dual arithmetic of Leibniz lay in the philosophical and religious mysticism associated with it. The 0 and 1, by which any number could be represented in that system, symbolized the creation of everything out of nothing; it afforded a phase of religious mystery which was thought to be helpful in the conversion of the heathen. The idea of Leibniz was based upon sound psychology. The mind of man delights in figures of speech, in analogies, in images. Here was an "*imago creationis*," truly novel and simple. The fact that it rests upon a mathematical basis was no drawback. Had not number-theory figured prominently in ancient religious mysticism?

With Leibniz his dual arithmetic was more than a passing fancy. He had reflected on this subject for over

twenty years, before he permitted an account of his meditations to appear in type. He made his first full statement of the binary scale and its symbolic interpretation in a letter written on January 2, 1697, to Duke Rudolph August of Brunswick. A little later, on May 17, 1698, Leibniz touched upon this subject in a letter to Johann Christian Schulenburg of Bremen, in which he states that his first thoughts on this matter antedate the year 1678. Accordingly his first ideas on binary numbers go back to the time when he was making his marvelous invention of the differential calculus. In April, 1701, Leibniz wrote enthusiastically on these numbers to John Bernoulli, then at Groningen in the Netherlands. Two years later, on July 12, 1703, he sent an account of his new arithmetic to Fontenelle, the secretary of the French Academy of Sciences, and it was published in 1703 under the title "Explication de l'arithmétique binaire," in the *Mémoires de l'Académie des Sciences de Paris*. This was the earliest appearance of this subject in print. The perusal of this article convinces the reader that Leibniz regarded it with parental pride.

A letter of Leibniz, written some years before, contains a statement which, we believe, has reference to the binary scale. It is a letter of September 8, 1690, sent to Placcius, who was professor of philosophy in the gymnasium in Hamburg. We may state parenthetically that this letter is of general interest, aside from its probable allusion to the binary scale. It reveals his ideas on the most profitable course of mathematical study and discloses information regarding the Hamburg mathematical club which ranks as the earliest organization of that sort known in mathematical history. We translate as follows:

"Recently I saw a book which deals in the German language with numerical problems, from which I gather that in Hamburg a few prominent arithmetical experts have

combined and formed a society with which others in that vicinity have become affiliated, and that Meissner, one of your countrymen and a teacher of arithmetic, is the leader of this movement. I am much pleased with this organization and I expect from it excellent things if they can make up their minds to expend their efforts upon matters which will enlarge the boundary of science; for to spend the time on special problems is not quite worthy of this undertaking, unless these problems are of particular elegance and usefulness or help to enlarge the field of the general method itself. Nothing is simpler than to collect problems which are easy for us who know the mode of procedure, but which cause others unnecessary labor. One should endeavor to perfect analysis itself, and I do not believe that there is any one in Germany who has acted in this matter with more zeal—not to say with greater success—than have I myself....

"I am also in possession of an invention for the construction of algebraic tables which, if once made available, would simplify computation and would afford to analysis almost as much aid as do the sine tables and logarithmic tables in ordinary arithmetic."

Does the last paragraph in this quotation refer to the binary system? In the letter written some years later (Jan. 2, 1697) to Duke Rudolph August, he says of this system:

"At the bottom of this there are so many wonderful things to see, useful also in the advancement of science, that some members of the Hamburg Arithmetical Society, whose diligence and aims are praiseworthy, could enjoyably direct their thoughts upon this and, as I can assure them, find things therein which would bring no little renown to them, and also to the German nation for having been first brought forth in Germany. For I see that from this mode of writ-

ing numbers there can be derived wonderful advantages profitably applicable also in ordinary arithmetic."

And what are the advantages which can be claimed for the binary system? In the first place it has no multiplication table beyond $1 \times 1 = 1$. Practically all operations can be performed by mere addition and subtraction. Consider for example the multiplication of 2 by 3. In the binary system $10 \times 11 = 110$, $11 \times 11 = (10 + 1)11 = 110 + 11 = 1001$. To be sure nearly four times as many figures must be written down in the binary scale as in the decimal scale, but the absence of a multiplication table is a vital gain. "Calculation as an effort of mathematical thought," says a recent writer, "might be said to be entirely dispensed with, and the labor of the brain to be all transferred to the eye and hand."

In his letter of Jan. 2, 1697, Leibniz accompanies his New Year's greetings to Duke Rudolph August by the remark: "That I shall not come this time altogether empty, I send you a symbol of what I recently had the honor to mention to you. It is in the form of a thought-penny or medal; and while my design is trifling and to be improved according to one's taste, yet the thing itself is of such a nature that it would seem worthy to be exhibited to posterity in silver, if such were to be stamped by the command of your gracious Highness. For one of the chief tenets of Christian faith, one of those which have met with the least acceptance on the part of the worldly wise and are not easily imparted to the heathen, relates to the creation of all things out of nothing by the all-power of God. It can be rightly claimed that nothing in the world better represents this, indeed almost proves it, than the origin of number in the manner represented here, where, by the use simply of unity and zero or nothing, all numbers originate. In nature and philosophy it will hardly be possible to find a better symbol of this mystery, for which reason

To render this medal, designed as an image of creation, still more attractive and artistic, Leibniz suggested that it should also represent light and darkness, the spirit of God moving upon the face of the waters. As a motto he chose the following:

"2, 3, 4, 5, etc. 0. *Omnibus ex nihilo ducendis sufficit unum.*" (To make all things from nothing, unity suffices.)

The binary arithmetic of Leibniz captured the attention of many mathematical writers. The mystic element put it in the class of mathematical recreations. Even Laplace, the heterodox, in his famous *Essai philosophique sur les probabilités,* speaks of it and its use in Chinese missions.

A curious blunder in mathematical history grew out of the binary arithmetic of Leibniz. The French Jesuit Bouvet, a missionary at Pekin and a zealous student of Chinese antiquities, learned of Leibniz's binary arithmetic and its theological interpretation. By the exercise of ingenious powers of coordination he found therein a key to the explanation of the *Cova,* or lineations of Fohi, the founder of the empire. They consisted of eight sets of three lines, either entire or broken lines, arranged in a circle. These *Cova* were held in great veneration in China, being suspended in all temples and, though not understood, were supposed to conceal great mysteries, embracing all true philosophy, both human and divine. Now Bouvet thought he had penetrated to the very depths of these mysteries when he announced triumphantly the discovery that in the *Cova* figures, the short lines meant 0 and the long lines meant 1, that Fohi possessed the principles of the binary arithmetic and that the *Cova* bore testimony to the unity of the Deity. Bouvet explained all this in a letter to Leibniz, dated Nov. 14, 1701. Leibniz, in turn, reported these findings in the paper to the Paris Academy which, as already related, was published in its *Mémoires* of 1703.

This application of the binary arithmetic to the interpretation of ancient oriental symbols afforded Leibniz profound pleasure. To the mathematician it meant the discovery that the Chinese had been in very early times in possession of binary arithmetic with its great principle of local value and the use of the zero. For the next 250 years the Chinese origin of this principle and of the zero appeared to be an established fact in mathematical history and was accepted as true even by the great mathematical historian of the nineteenth century, Moritz Cantor of Heidelberg, in his earlier publications. However, in 1863 Cantor became convinced that the traditional interpretation was incorrect, that the *Covas* of Fohi are not numbers at all, but have a physical significance, representing, respectively, air, rain, water, mountain, earth, thunder, fire, wind.

Thus it is seen that Leibniz's very minor invention of dual arithmetic was to him an object of contemplation for over a quarter of a century; it afforded him a satisfaction out of all proportion to its importance. He corresponded on the subject with mathematicians and religious teachers. It gave rise to an interesting chapter in modern religious mysticism and in the annals of foreign missions; it led to a blunder in the history of numeral notations which persisted for two centuries and a half, until the time of a great mathematical historian who is still living. It was the point of departure of interesting speculations as to the relative advantages of numeral notations whose bases are powers of 2, that is, the bases 2, 4 and 8.

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LEIBNIZ'S MONADS AND BRADLEY'S FINITE CENTERS.

NO philosopher is more fantastic than Leibniz in presentation, few have been less intelligently interpreted. At first sight, none is less satisfactory. Yet Leibniz remains to the end disquieting and dangerous. He represents no one tradition, no one civilization; he is allied to no social or literary tendency; his thought cannot be summed up or placed. Spinoza represents a definite emotional attitude; suggestive as he is, his value can be rated. Descartes is a classic, and is dead. "Candide" is a classic: Voltaire was a wise man, and not dangerous. Rousseau is not a classic, nor was he a wise man; he has proved an eternal source of mischief and inspiration. Reviewing the strange opinions, almost childish in *naiveté,* of birth and death, of body and soul, of the relation between vegetable and animal, of activity and passivity—together with the pitiful efforts at orthodoxy and the cautious ethics of this German diplomat, together with his extraordinary facility of scientific insight, one is disconcerted at the end. His orthodoxy is more alarming than others' revolution, his fantastic guesses more enduring than others' rationality.

Beside the work of Russell and of Couturat I have found only one author of assistance in attempting to appreciate the thought of Leibniz. In Bradley's *Appearance and Reality* I seemed to find features strikingly similar to those of monadism. So that re-reading Leibniz I cannot

help thinking that he was the first to express, perhaps half unconsciously, one of those fundamental varieties of view which perpetually recur as novelties. With his motives, logical and otherwise, I am not here concerned. I only wish to point out, and leave for consideration, certain analogies.

That monadism begins with Leibniz I think will be conceded. It is characteristic of the man that everything about his monads, except the one essential point which makes them his own, he may have borrowed from an author with whom he was certainly acquainted. Bruno's theory has everything in common with that of Leibniz except this one point. A kind of pre-established harmony, the continuity of animal and vegetable and of organic and inorganic, the representation of the whole in the part, even the words *monadum monas*: these points of identity one finds.[1] But the monad of Bruno has this difference: it has windows. And it is just the impenetrability of the Leibnizian monads which constitutes their originality and which seems to justify our finding a likeness between Leibniz and Bradley. In any case, there is no philosopher with whom the problem of sources is less important than with Leibniz. The fact that he could receive stimulation from such various sources and remain so independent of the thought of his own time[2] indicates both the robustness and the sensitiveness of genius. He has studied Thomas, and probably with great care the *Metaphysics* and the

[1] See H. Brunnhofer, *G. Bruno's Lehre vom Kleinsten als die Quelle der praestabilierten Harmonie von Leibniz* (Leipsic, 1890), for quotations, e. g.: *De trip. min.*: "Deus est monadum monas." Also *Spaccio della bestia trionfante*: "In ogni uomo, in ciascuno individuo si contempla un mondo, un universo." Brunnhofer even traces the window metaphor back to the Song of Solomon: "Prospiciens per fenestras."

[2] At least he affirms his independence. In 1679 he writes to Malebranche that as when he began to meditate he was not imbued with Cartesian opinions, he was led to "entrer dans les choses par une autre porte et decouvrir de nouveaux pays." He is also inclined to speak rather slightingly of Spinoza. See Wendt, *Die Entwickelung der Leibnizischen Monadenlehre bis zum Jahre 1695* (Berlin, 1886). The germs of monadism appear as early as 1663.

De anima, but he is not an Aristotelian; he was probably profoundly struck by the passage *Sophistes* 247e, but any one who has read his panegyric of the Phaedo (*Discourse,* XXVI) will probably agree that his praise is more the approval of posterity than the interpretation of discipleship. Leibniz's originality is in direct, not inverse ratio to his erudition.

More than multiplicity of influences, perhaps the multiplicity of motives and the very occasional reasons for some of Leibniz's writings, make him a bewildering and sometimes ludicrous writer. The complication of his interests in physics, his interests in logic, and his equally genuine interest in theology, make his views a jungle of apparent contradictions and irrelevancies. His theory of physical energy, for example, leads to an unsound metaphysical theory of activity, and his solicitude for the preservation of human immortality leads to a view which is only an excrescence upon monadism,[3] and which is in every way less valuable than Aristotle's. Thus there are features of the theory which are inessential. When we confine our attention to the resemblances between Leibniz's and Bradley's views, we will find I think that they cover everything essential. These are (1) complete isolation of monads from each other; (2) sceptical theory of knowledge, relativistic theory of space, time, and relations, a form of anti-intellectualism in both writers; from which follows (3) the indestructibility of the monads; (4) the important doctrine of "expression."[4] Certain distinctions of Bradley's, as the (relative) distinction between finite centers and selves, are also implicit in Leibniz. The relation of soul and body, the possibility of pan-psychism, the knowledge of soul by soul, are problems which come to closely similar solutions in the two philosophies.

[3] It leads Leibniz almost to the admission that persistence in the case of the lower types of monad is meaningless. Cf. *Discourse,* XXXIV.

[4] See Letter to Arnauld, Oct. 6, 1687.

I suggest that from the "pluralism" of Leibniz there is only a step to the "absolute zero" of Bradley, and that Bradley's Absolute dissolves at a touch into its constituents.

In the first place, Leibniz's theory of degrees of perfection among monads approximates to a theory of degrees of reality. Mr. Russell has pointed out how easy a step it would have been for Leibniz to have made reality the subject of all predicates. The world consists of simple substances and their states. The subject is never, even from a timeless point of view, merely equivalent to the sum of its states; it is incapable of exhaustion by any addition of predicates. The question with which Leibniz attempted to cope in his first thesis, and the question which he was never able satisfactorily to settle, was what makes a real subject, what the principle of individuation is. Nowhere in the correspondence with Arnauld do we find a trustworthy mark of differentiation between substantial and accidental unities. If everything which can have predicates, everything which can be an object of attention is a substance, the whole theory falls to the ground; but if this is not the case, we shall either be obliged to make reality the subject of all predicates, or we shall be forced to distinguish, as do some idealists, between judgments and pseudo-judgments, and the logical basis for monadism fails. If we cannot find by inspection an obvious and indubitable token of difference between the substantial and the accidental, we shall in the end find substantiality only in reality itself; or, what comes to the same thing, we shall find degrees of substantiality everywhere. In the latter case substance becomes relative to finite and changing points of view, and in the end again we must seek refuge in the one substance, or resign ourselves to find no refuge at all.

This omnipresence of substance, in degree, comes very near at times to being Leibniz's true doctrine. "One thing

experience is common to all; and it is by standing on this basis that we are able to communicate. Such a statement would be incorrect. My external sensations are no less private to myself than are my thoughts or my feelings. In either case my experience falls within my own circle, a circle closed on the outside; and with all the elements alike, every sphere is opaque to the others which surround it. With regard to communicability, there is in fact not any difference of kind, but only of degree...It is not true that our physical experiences have unity in any sense which is inapplicable to the worlds we call internal...In brief, regarded as existence which appears in a soul, the whole world for each is peculiar and private to that soul... No experience can lie open to inspection from outside; no direct guarantee of identity is possible...That real identity of ideal content, by which all souls live and move, cannot work in common save by the paths of external appearance."[9]

Perhaps this is only a statement of a usual idealistic position, but never has it been put in a form so extreme. A writer to whose words Mr. Bradley would probably subscribe, Professor Bosanquet, formulates the orthodox view: "No phase in a particular consciousness is merely a phase in that consciousness, but it is always and essentially a member of a further whole of experience, which passes through and unites the states of many consciousnesses."[10] This view Mr. Bradley also holds. But he more often emphasizes the other aspect. Each finite center is, "while it lasts," the whole world. The world of practice, the world of objects, is constructed out of the ideal identities intended by various souls.

For Bradley, I take it, an object is a common intention of several souls, cut out (as in a sense are the souls them-

[9] *Appearance*, p. 343ff.
[10] *Principle of Individuality and Value*, p. 315.

selves) from immediate experience. The genesis of the common world can only be described by admitted fictions, since in the end there is no question of its origin in time: on the one hand our experiences are similar because they are of the same objects, and on the other hand the objects are only "intellectual constructions" out of various and quite independent experiences. So, on the one hand, my experience is in principle essentially public. My emotions may be better understood by others than by myself; as my oculist knows my eyes. And on the other hand everything, the whole world, is private to myself. Internal and external are thus not adjectives applied to different contents within the same world; they are different points of view.

I will pass now to another consideration. Is the finite center or the soul the counterpart to the monad? It is very difficult to keep the meanings of "soul," "finite center," and "self" quite distinct. All are more or less provisional and relative. A self is an ideal and largely a practical construction, one's own self as much as that of others. My self "remains intimately one thing with that finite center within which my universe appears. Other selves on the contrary are for me ideal objects."[11] The self is a construction in space and time. It is an object among others, a self among others, and could not exist save in a common world. The soul (as in the passage quoted at length) is almost the same as finite center. The soul, considered as finite center,[12] cannot be acted upon by other entities, since a finite center is a universe in itself." "If you confine your attention to the soul as a soul, then every possible experience is more than what happens in and to this soul. You have to do with psychical events which qualify the soul, and in the end these events, so far as you are true to your idea, are merely states of the soul. Such

[11] *Truth and Reality*, p. 418.

[12] "A soul is a finite center viewed as an object existing in time with a before and after of itself," *ibid.*, p. 414.

a conception is for certain purposes legitimate and necessary..."[13] Change, accordingly, cannot be due to any agency outside of these states themselves; it can only be, "in every state of a substance, some element or quality in virtue of which that state is not permanent, but tends to pass into the next state. This element is what Leibniz means by activity."[14]

The soul only differs from the finite center in being considered as something not identical with its states. The finite center so far as I can pretend to understand it *is* immediate experience. It is not in time, though we are more or less forced to think of it under temporal conditions. "It comes to itself as all the world and not as one among others. And it has properly no duration through which it lasts. It can contain a lapse and a before and after, but these are subordinate."[15] The finite center in a sense contains its own past and future. "It has, or it contains, a character, and on that character its own past and future depend."[16] This is more clearly the case with the soul. But it would be untrue to go on and declare that the soul "bears traces" of everything that happens to it. It would be a mistake to go on, holding this view of the soul, and distinguish between various grades of soul according to faculty. This would be to confuse the soul which is a whole world, to which nothing comes except as its own attribute and adjective, with the soul which can be described by its way of acting upon an environment. In this way Leibniz thrusts himself into a nest of difficulties. The concepts of center, of soul, and of self and personality must be kept distinct. The point of view from which each soul is a world in itself must not be confused with the point of view from which each soul is only the function of a physical organism, a unity perhaps only par-

[13] *Ibid.*, p .415.
[14] Russell, p. 44.
[15] *Truth and Reality*, p. 410.
[16] *Ibid.*, p. 411.

tial, capable of alteration, development, having a history and a structure, a beginning and apparently an end. And yet these two souls are the same. And if the two points of view are irreconcilable, yet on the other hand neither would exist without the other, and they melt into each other by a process which we cannot grasp.] If we insist upon thinking of the soul as something *wholly* isolated, as *merely* a substance with states, then it is hopeless to attempt to arrive at the conception of other souls. For if there are other souls, we must think of our own soul as more intimately attached to its own body than to the rest of its environment; we detach and idealize some of its states. We thus pass to the point of view from which the soul is the entelechy of its body. It is this transition from one point of view to another which is known to Mr. Bradley's readers as transcendence. It is the failure to deal adequately with transcendence, or even to recognize the true nature of the problem, which makes Leibniz appear so fantastic, and puts him sometimes to such awkward shifts.

Thus Leibniz, while he makes the soul the entelechy of the body, is forced to have recourse to the theory of the dominant monad. Now I contend that if one recognizes two points of view, which are irreconcilable and yet melt into each other, this theory is quite superfluous. It is really an attempt to preserve the reality of the external world at the same time that it is denied, which is perhaps the attempt of all pan-psychism: to substitute for two concepts which have at least a relative validity in practice—consciousness and matter—one which is less useful and consequently less significant, animated matter. So far as my body is merely an adjective of my soul I suppose that it needs no outside explanation; and so far as it possesses an independent reality it is quite unnecessary to say that this is because it is compounded of elements

which are adjectives of other souls or monads. Leibniz has here done no more than to add to the concepts of psychical and physical a third and otiose concept.

The monad in fact combines, or attempts to combine, several points of view in one. Because Leibniz tries to run these different aspects together, and at the same time refuses to recognize that the independence and isolation of the monads is only a relative and partial aspect, he lets himself in for the most unnecessary of his mysteries—the pre-established harmony. Bradley turns the Absolute to account for the same purpose. "The one Absolute" knows itself and realizes itself in and through finite centers. "For rejecting a higher experience," Mr. Bradley says, "in which appearances are transformed, I can find no reason."[17] But what we do know is that we are able to pass from one point of view to another, that we are compelled to do so, and that the different aspects more or less hang together. For rejecting a higher experience there may be no reason. But that this higher experience explains the lower is at least open to doubt.

Mr. Bradley's monadism is in some ways a great advance beyond Leibniz's. Its technical excellence is impeccable. It unquestionably presents clearness where in Leibniz we find confusion. I am not sure that the ultimate puzzle is any more frankly faced, or that divine intervention plays any smaller part. Mr. Bradley is a much more skilful, a much more finished philosopher than Leibniz. He has the melancholy grace, the languid mastery, of the late product. He has expounded one type of philosophy with such consummate ability that it will probably not survive him. In Leibniz there are possibilities. He has the permanence of the pre-Socratics, of all imperfect things.

LONDON, ENGLAND. T. STEARNS ELIOT.

[17] *Truth and Reality*, p. 413.

THE MANUSCRIPTS OF LEIBNIZ ON HIS DISCOVERY OF THE DIFFERENTIAL CALCULUS.

APART from the intrinsic interest which the autograph writings, and more particularly the earlier efforts, of any of the prime movers in any branch of learning possess for the historical student, there is a special interest attached to the manuscripts and correspondence of Leibniz. They are invaluable as an aid to the study of the part that their author played in the invention and development of the infinitesimal calculus. More especially is this so in the case of Leibniz; for the matter, upon which this essay is founded, unearthed by Dr. C. I. Gerhardt in a mass of papers belonging to Leibniz that had been preserved in the Royal Library of Hanover, contained holographs previously unpublished.

The most important of these, for our purpose, were edited, with full notes and a commentary, by Gerhardt, in three separate volumes, under the respective titles:[1]

1. *Historia et Origo Calculi Differentialis, a G. G. Leibnizio conscripta.* Hanover, 1846.
2. *Die Entdeckung der Differentialrechnung durch Leibniz.* Halle, 1848.
3. *Die Geschichte der höheren Analysis; erste Abtheilung, Die Entdeckung der höheren Analysis.* Halle, 1855.

[1] Because of the length and mathematical character of many of the footnotes to the Leibniz translations it has been found necessary to have them follow consecutively after the text. See "Notes," page 611.

The present time, the two hundredth anniversary of the death of Leibniz, would seem to be a most suitable one for publishing an English translation of these manuscripts.

For the present purpose, it will be convenient to group the manuscripts in two sections, of which the first will consist of Leibniz's own account of his work. Under the heading § I below is given a fairly literal translation of a postscript from Leibniz to Jakob (i. e., James) Bernoulli, "which was written from Berlin in April 1703, and then cancelled and a postscript on a totally different subject substituted."[2] This is a communication to a more or less intimate friend. It is therefore naturally not such a considered composition as the second account that Leibniz gives of his work in the *Historia* mentioned above, of which a full translation is given below under the heading § II. It is important to bear this point in mind when comparing the two accounts together, for any slight discrepancies that may be noticed are, feasibly at least, to be accounted for by the different circumstances of the compositions. The latter account bears the impress of being fairly fully revised and made ready for press, and the facts marshalled to make an impressive or, as some would have it, plausible whole; it was probably finished just before the death of Leibniz, and represents his answer to the *Commercium Epistolicum* of unsavory memory. The death of Leibniz in November 1716 was probably the cause which prevented its publication, or at least the chief reason.

It is not my intention to enter into a discussion about the *Commercium Epistolicum*; this has probably had the last word said upon it that it is possible to say with the help of the existing authentic material that is possessed by the present-day historians of mathematics. Further, I hold quite other views as to the possible source of Leibniz's inspiration, if indeed he is not to be credited with

perfectly independent discovery. I will therefore, as far as I may, refrain from allusion to the *Commercium Epistolicum,* except to second the plea of its perfectly disgraceful unfairness, as made by Leibniz.[3] I have suggested above that the *Historia* was intended by Leibniz as a statement of his side of the case, and as an answer to the attack made upon him. This account of his work, although written in the third person, "by a friend who knew all about the matter,"[4] is, on the authority of Gerhardt, undoubtedly by Leibniz himself. Without in any way impugning this authority, I cannot help thinking it would have been more satisfactory if I could have included herein photographic copies of parts of this manuscript; but this is impossible at the time of writing.

The reasons for the delay in the preparation of the *Historia* are stated in the manuscript itself; and later I shall have occasion to discuss these. In order that the remarks made may in all cases be perfectly intelligible, I must here give a very short account[5] of the history of the quarrel up to the time of the publication of the *Commercium Epistolicum* in 1712.

The matter was first started in the year 1699 by Fatio de Duillier, a Swiss mathematician who had been living in London since 1691; he was a correspondent of Huygens, and from letters that Fatio sent to Huygens[6] it would appear that the attack had been quietly in preparation for some time. Whether he had Newton's sanction or not cannot be ascertained, yet it seems certain from the correspondence that Newton had given Fatio information with regard to his writings. Fatio then concludes that Newton is the first discoverer and that Leibniz, as second discoverer, has borrowed from Newton. These accusations hurt Leibniz all the more, because he had deposited copies of his correspondence with Newton in the hands of Wallis for publication. As Fatio was a member of the

Royal Society, Leibniz took it for granted that Fatio's attack was with the approval of that body; he asked therefore that the papers in the hands of Wallis should be published in justice to himself. He received a reply from Sloane, one of the secretaries of the Society, informing him that his assumption with regard to any such participation of the Society in the attack was groundless; and in consequence of this he took no further notice of the matter, and the whole thing lapsed into oblivion.

In the year 1708 the attack against Leibniz was renewed by Keill; and the charge that Leibniz had borrowed from Newton was most directly made. Leibniz had nobody in England who was in a position to substantiate his claims, for Wallis had died in 1703; so he appealed directly to the Royal Society. This body in consequence appointed a commission composed of members of the Society to consider the papers concerned in the matter. Their report appeared in the year 1712 under the title of *Commercium Epistolicum D, Johannis Collins et Aliorum de Analysi promota, jussu Societatis Regiae in lucem editum.*

Leibniz did not return to Hanover, from a tour of the towns of Italy on genealogical research work, until two years later; so that the date of the *Historia* is definitely established to have between 1714 and 1716, the date of his death. The dates allow us to account for the similarity between the two reports he gives of his work, in the postscript and the *Historia,* and also for any slight discrepancies between them.

Let us first, however, try to find a reason why the postscript was written, and having been written why it was cancelled. In the *Acta Eruditorum* (Leipsic) for January 1691, James Bernoulli said that Leibniz had got his fundamental ideas from Barrow;[7] but in a later number, that for June 1691, he admitted that Leibniz was far in advance of Barrow, though both views were alike in some ways.[8] One

is inclined to wonder whether this admission was a result of Leibniz's reputed personality and charm; but as Leibniz seems to have been stationed at Wolfenbüttel and Bernoulli at Basel at this time a personal interview would seem improbable, and a more feasible suggestion would seem to be a reasoned remonstrance by letter from Leibniz. It is to be noticed that Bernoulli does not exactly retract his statement that Leibniz had Barrow to thank for the fundamental ideas, he only states that in spite of the similarities there are also dissimilarities in which Leibniz stands far above Barrow.[9] I am inclined to think he is simply comparing the method of Leibniz with the differential triangle method of Barrow, and that Bernoulli even has not noticed that Barrow has propositions that are the geometrical equivalents of the differentiation of a product, quotient and powers of the *dependent* variables.

It seems to me that at this time Leibniz, though he does not forget his insinuation, has to lay all thoughts of combating it aside; for Gerhardt apparently found no other letters or other manuscripts referring to the matter prior to that of 1703. At a certain time later, judging from the first paragraph of the intended postscript, he would appear to have referred to the matter again, and to have called forth from the Bernoullis an excuse or a justification of the statements in the *Acta Eruditorum,* together with some expression of their surprise that he should have been upset over them. The reason may have been that it got to the ears of Leibniz that the opinion was not confined to the Bernoullis, for Leibniz says "....you, your brother, *or any one else.*"[10]

Thus much we may guess as to the occasion that prompted the writing of the postscript; now let us try to find the reason for its being cancelled. Fatio's attack seems to have been precipitated through pique at having been left out by Leibniz in a list of mathematicians alone capable

of solving John Bernoulli's problem of the line of quickest descent.[11] "He published a memoir on the problem, in which he declared that he was obliged by the undeniable evidence of things to acknowledge Newton, not only as the first, but as by many years the first, inventor of the calculus; from whom, whether Leibniz, the second inventor, borrowed anything or not, he would rather *those who had seen Newton's letters and other manuscripts should judge than himself.*" The attack in itself is cowardly, in that Fatio does not dare to make a direct assertion, only an insinuation that is far more damaging, since it suggests that to those who have seen the papers of Newton the matter could not be in the slightest doubt. Leibniz replied by an article in the *Acta Eruditorum,* for May 1700, in which he cited Newton's letters, as also the testimony which Newton rendered to him in the *Principia,*[12] as proof of his claim to an independent authorship of his method. A reply was sent by Duillier, which the editors of the *Acta Eruditorum* refused to publish. This would probably be in 1701; and I suggest that Leibniz had probably now come to the conclusion that it would be wiser to let the matter of Barrow drop and attend to the affair with Newton. When he, unwisely, started the controversy once more by a review (containing what was taken to be an implied sneering allusion to Newton) of the *Tractatus de Quadratura Curvarum,* published by the latter with his Optics in 1708,[13] and thus drew upon himself the attack by Keill, he gladly allowed the suggestion about Barrow to fade into oblivion, cast out by the more public, but I think the less true, charge of plagiarism from Newton. He also saw that he would have to prepare a careful answer if he made one at all, and second thoughts suggested that it would be as well if his postscript was made the matter for further consideration, correction, if necessary, and amplification, before it was sent off. It is to be noted

that the review above mentioned is written anonymously in the third person, but it has been established that its author was Leibniz himself.[14]

There does not seem to be any occasion for further general remarks; particular points of criticism will be alluded to as the translation given below proceeds.

PART I.

§ I.

Full translation of the intended postscript to the letter to James Bernoulli, dated April 1703, from Berlin.

Perhaps[15] you will thing it small-minded of me that I should be irritated with you, your brother, or any one else, if you should have perceived the opportunities for obligation to Barrow, which it was not necessary for me, his contemporary[16] in these discoveries, to have obtained from him.

When I arrived in Paris in the year 1672, I was self-taught as regards geometry,[17] and indeed had little knowledge of the subject, for which I had not the patience to read through the long series of proofs. As a youth I consulted the beginner's Algebra of a certain Lanzius,[18] and afterward that of Clavius;[19] that of Descartes seemed to be more intricate.[20] Nevertheless, it seemed to me, I do not know by what rash confidence in my own ability, that I might become the equal of these if I so desired. I also had the audacity to look through even more profound works, such as the geometry of Cavalieri,[21] and the more pleasant elements of curves of Leotaud,[22] which I happened to come across in Nuremberg, and other things of the kind; from which it is clear that I was now ready to get along without help,[23] for I read them almost as one reads tales of romance.

Meanwhile I was fashioning for myself a kind of geometrical calculus by means of little squares and cubes to express undetermined numbers, being unaware that Descartes and Vieta had worked out the whole matter in a superior manner.[24] In this, I may almost call it, superb ignorance of mathematics, I was then studying history and law; for I had decided to devote myself to the latter.

From mathematics I as it were only sipped those things that were the more pleasant, being especially fond of investigating and inventing machines, for it was at this time that my arithmetical machine[25] was devised. At this time also it happened that Huygens, who I fully believe saw more in me than there really was, with great courtesy brought me a copy recently published of his book on the pendulum.[26] This was for me the beginning or occasion of a more careful study of geometry. While we conversed, he perceived that I had not a correct notion of the center of gravity, and so he briefly described it to me; at the same time he added the information that Dettonville (i. e., Pascal) had worked such things out uncommonly well.[27] Now I, who always had the peculiarity that I was the most teachable of mortals, often cast aside innumerable meditations of mine that were not brought to maturity, when as it were they were swallowed up in the light shed upon them by a few words from some great man, immediately to grasp with avidity the teachings of a mathematician of the highest class; for I quickly saw how great was Huygens. In addition there was the stimulus of shame, in that I appeared to be ignorant with regard to such matters. So I sought a Dettonville from Buotius, a Gregory St. Vincent[28] from the Royal Library, and started to study geometry in earnest. Without delay I examined with delight the *"ductus"* of St. Vincent, and the *"ungulae"*[29] begun by St. Vincent and developed by Pascal, and those sums and sums of sums and solids formed and resolved in various ways; for they afforded me more pleasure than trouble.

I was working upon these when I happened to come across a proof of Dettonville's that was of a supremely easy nature, by which he proved the mensuration of the sphere as given by Archimedes,[30] and showed from the similarity of the triangles EDC and CBK that CK into DE = BC into EC; and hence, by taking BF = CK, that the rectangle AF is equal to the moment[31] of the curve AEC about the axis AB. [Fig. 1.]

The novelty of the reasoning struck me forcibly, for I had not noticed it in the works of Cavalieri.[32] But nothing astonished me so much as the fact that Pascal seemed to have had his eyes obscured by some evil fate; for I saw at a glance that the theorem was a most general one for any kind of curve whatever. Thus, let the perpendiculars not all meet in a point, but let each perpendicular from the curve be transferred to the position of an ordinate to the axis, as

PC or (P)(C) to the position BF or (B)(F); then it is clear that the zone FB(B)(F)F will be equal to the moment of the curve C(C) about the axis.[33] [Fig. 2.]

Straightway I went to Huygens, whom I had not seen again in the meantime. I told him that I had followed out his instructions and that I was now able to do something that Pascal had failed to do. Then I showed him the general theorem for moments of curves. He was struck with wonder and said, "Now, that is the very theorem upon which depend my constructions for finding the area[34] of the surfaces of parabolic, elliptic and hyperbolic conoids; and how these were discovered, neither Roberval nor Bullialdus[35] were ever able to understand." Thus praising my progress, he asked me whether I could not now find the properties of such curves as F(F).

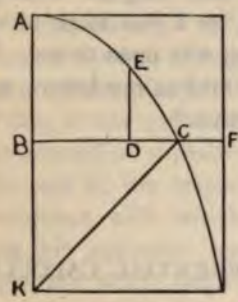


Fig. 1.

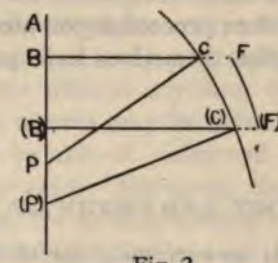


Fig. 2.

When I told him that I had made no investigation in this direction he told me to read the works of Descartes and Slusius,[36] who showed how to form equations for loci; for he said that this idea was a most useful one. Thereupon I examined the Geometry of Descartes and made a close study of Slusius, thus entering the house of geometry truly as it were by the back door. Urged on by the success I met with, and by the great number of results that I obtained, I filled some hundreds of sheets with them in that year. These I divided into two classes of assignables and inassignables. Among assignables I placed everything I obtained by the methods previously used by Cavalieri, Guldinus, Toricelli, Gregory St. Vincent and Pascal, such as sums, sums of sums, transpositions, "ductus," cylinders truncated by a plane, and lastly by the method of the center of gravity; and among inassignables I placed all that I obtained by the use of the triangle which I at that time called "the

characteristic triangle,"[37] and things of the same class, of which Huygens and Wallis seemed to me to have been the originators.

A little later there fell into my hands the Universal Geometry of James Gregory of Scotland,[38] in which I saw the same idea exploited (although obscured by the proofs, which he gave according to the manner of the ancients), and as in Barrow, when his Lectures appeared, in which latter I found the greater part of my theorems anticipated.[39]

However I did not mind this very much, since I saw that these things were perfectly easy to the veriest beginner who had been trained to use them,[40] and because I perceived that there remained much higher matters, which however required a new kind of calculus. Thus I did not think that my Arithmetical Quadrature, although it was received by the French and English with great commendation, was worth being published, as I was loath to waste time over such trifles while the whole ocean was open to me. How matters then proceeded you already know, and as my letters, which the English themselves have published, prove.[41]

§ II.

HISTORY AND ORIGIN OF THE DIFFERENTIAL CALCULUS.

It is an extremely useful thing to have knowledge of the true origins of memorable discoveries, especially those that have been found not by accident but by dint of meditation. It is not so much that thereby history may attribute to each man his own discoveries and that others should be encouraged to earn like commendation, as that the art of making discoveries should be extended by considering noteworthy examples of it.

Among the most renowned discoveries of the times must be considered that of a new kind of mathematical analysis, known by the name of the differential calculus; and of this, even if the essentials are at the present time considered to be sufficiently demonstrated, nevertheless the origin and the method of the discovery are not yet known to the world at large. Its author invented it nearly forty years ago, and nine years later (nearly thirty years ago) published it in a concise form; and from that time it has not only been frequently made known in memoirs,[42] but also has been a method of general employment; while many splendid discoveries

have been made by its assistance, such as have been included in the *Acta Eruditorum*, Leipsic, and also such as have been published in the memoirs of the Royal Academy of Sciences; so that it would seem that a new aspect has been given to mathematical knowledge arising out of its discovery.

Now there never existed any uncertainty as to the name of the true inventor, until recently, in 1712, certain upstarts, either in ignorance of the literature of the times gone by, or through envy, or with some slight hope of gaining notoriety by the discussion, or lastly from obsequious flattery, have set up a rival to him; and by their praise of this rival, the author has suffered no small disparagement in the matter, for the former has been credited with having known far more than is to be found in the subject under discussion. Moreover, in this they acted with considerable shrewdness, in that they put off starting the dispute until those who knew the circumstances, Huygens, Wallis, Tschirnhaus, and others, on whose testimony they could have been refuted, were all dead.[43] Indeed this is one good reason why contemporary prescripts should be introduced as a matter of law; for without any fault or deceit on the part of the responsible party, attacks may be deferred until the evidence with which he might be able to safeguard himself against his opponent had ceased to exist. Moreover, they have changed the whole point of the issue, for in their screed, in which under the title of *Commercium Epistolicum D. Johannis Collinsii* (1712) they have set forth their opinion in such a manner as to give a dubious credit to Leibniz, they have said very little about the calculus; instead, every other page is made up of what they call infinite series. Such things were first given as discoveries by Nicolaus Mercator[44] of Holstein, who obtained them by the process of division, and Newton gave the more general form by extraction of roots.[45] This is certainly a useful discovery, for by it arithmetical approximations are reduced to an analytical reckoning; but it has nothing at all to do with the differential calculus. Moreover, even in this they make use of fallacious reasoning; for whenever this rival works out a quadrature by the addition of the parts by which a figure is gradually increased,[46] at once they hail it as the use of the differential calculus (as for instance on page 15 of the *Commercium*). By the selfsame argument, Kepler (in his *Stereometria Doliorum*),[47] Cavalieri, Fermat, Huygens, and Wallis used the differential calculus; and indeed, of those who dealt with "in-

divisibles" or the "infinitely small," who did not use it? But Huygens, who as a matter of fact had some knowledge of the method of fluxions as far as they are known and used, had the fairness to acknowledge that a new light was shed upon geometry by this calculus, and that knowledge of things beyond the province of that science was wonderfully advanced by its use.

Now it certainly never entered the mind of any one else before Leibniz to institute the notation peculiar to the new calculus by which the imagination is freed from a perpetual reference to diagrams, as was made by Vieta and Descartes in their ordinary or Apollonian geometry; moreover, the more advanced parts pertaining to Archimedean geometry, and to lines which were called "mechanical"[48] by Descartes, were excluded by the latter in his calculus. But now by the calculus of Leibniz the whole of geometry is subjected to analytical computation, and those transcendent lines that Descartes called mechanical are also reduced to equations chosen to suit them, by considering the differences dx, ddx, etc., and the sums that are the inverses of these differences, as functions of the x's; and this, by merely introducing the calculus, whereas before this no other functions were admissible but x, xx, x^3, $\sqrt{x}$, etc., that is to say, powers and roots.[49] Hence it is easy to see that those who expressed these differences by 0, as did Fermat, Descartes, and even that rival, in his *Principia* published in 16—,[50] were by that very fact an extremely long way off from the differential calculus; for in this way neither gradation of the differences nor the differential functions of the several quantities can possibly be made out.

There does not exist anywhere the slightest trace of these methods having been practised by any one before Leibniz.[51] With precisely the same amount of justice as his opponents display in now assigning such discoveries to Newton, any one could equally well assign the geometry of Descartes to Apollonius, who, although he possessed the essential idea of the calculus, yet did not possess the calculus.

For this reason also the new discoveries that were made by the help of the differential calculus were hidden from the followers of Newton's method, nor could they produce anything of real value nor even avoid inaccuracies until they learned the calculus of Leibniz, as is found in the investigation of the catenary as made by David Gregory.[52] But these contentious persons have dared to misuse the name of the English Royal Society, which body took

pains to have it made known that no really definite decision was come to by them; and this is only what is worthy of their reputation for fair dealing, in that one of the two parties was not heard, indeed my friend himself did not know that the Royal Society had undertaken an inquiry into the matter. Else the names of those to whom it had entrusted the report would have been communicated to him,[53] so that they might either be objected to, or equipped for their task. He indeed, astounded not by their arguments but by the fictions that pervaded their attack on his good faith, considered such things unworthy of a reply, knowing as he did that it would be useless to defend his case before those who were unacquainted with this subject (i. e., the great majority of readers); also feeling that those who were skilled in the matter under discussion would readily perceive the injustice of the charge.[54] To this was added the reason that he was absent from home when these reports were circulated by his opponents, and returning home after an interval of two years and being occupied with other business, it was then too late to find and consult the remains of his own past correspondence from which he might refresh his memory about matters that had happened so long ago as forty years previously. For transcripts of very many of the letters once written by him had not been kept; besides those that Wallis found in England and published with his consent in the third volume of his works, Leibniz himself had not very many.

Nevertheless, he did not lack for friends to look after his fair name; and indeed a certain mathematician, one of the first rank of our time[55] well skilled in this branch of learning and perfectly unbiased, whose good-will the opposite party had tried in vain to obtain, plainly stated, giving reasons of his own finding, and let it be known, not altogether with strict justice, that he considered that not only had that rival not invented the calculus, but that in addition he did not understand it to any great extent.[56] Another friend of the inventor[57] published these and other things as well in a short pamphlet, in order to check their base contentions. However it was of greater service to make known the manner and reasoning by which the discoverer arrived at this new kind of calculus; for this indeed has been unknown up till now, even to those perchance, who would like to share in this discovery. Indeed he himself had decided to explain it, and to give an account of the course of his researches in analysis partly from memory and partly from extant

writings and remains of old manuscripts, and in this manner to illustrate in due form in a little book the history of this higher learning and the method of its discovery. But since at the time this was found to be impossible owing to the necessities of other business, he allowed this short statement of part of what there was to tell upon the matter to be published in the meantime by a friend who knew all about it,[58] so that in some measure public curiosity should be satisfied.

The author of this new analysis, in the first flower of his youth, added to the study of history and jurisprudence other more profound reflections for which he had a natural inclination. Among the latter he took a keen delight in the properties and combinations of numbers; indeed, in 1666 he published an essay, *De Arte Combinatoria*, afterward reprinted without his sanction. Also, while still a boy, when studying logic he perceived that the ultimate analysis of truths that depended on reasoning reduced to two things, definitions and identical truths, and that these alone of the essentials were primitive and undemonstrable. When it was stated in contradiction that identical truths were useless and nugatory, he gave illustrative proofs to the contrary. Among these he gave a demonstration that that mighty axiom, "The whole is greater than its part," could be proved by a syllogism of which the major term was a definition and the minor term an identity.[59] For if one of two things is equal to a part of another the former is called the less, and the later the greater; and this is to be taken as the definition. Now, if to this definition there be added the following identical and undemonstrable axiom, "Every thing possessed of magnitude is equal to itself," i. e., A = A, then we have the syllogism:

Whatever is equal to a part of another, is less than that other:
(by the definition)

But the part is equal to a part of the whole:
(i. e., to itself, by identity)

Hence the part is less than the whole. Q. E. D.

As an immediate consequence of this he observed that from the identity A = A, or at any rate from its equivalent, A − A = 0, as may be seen at a glance by straightforward reduction, the following very pretty property of differences arises, namely:

$$A \quad \underbrace{-A+B} \quad \underbrace{-B+C} \quad \underbrace{-C+D} \quad \underbrace{-D+E} \quad -E = 0$$
$$+ \quad L \quad + \quad M \quad + \quad N \quad + \quad P$$

If now A, B, C, D, E are supposed to be quantities that continually increase in magnitude, and the differences between successive terms are denoted by L, M, N, P, it will then follow that

$$A+L+M+N+P=0,$$
$$\text{i. e.,} \quad L+M+N+P=E-A;$$

that is, the sums of the differences between successive terms, no matter how great their number, will be equal to the difference between the terms at the beginning and the end of the series.[60] For example, in place of A, B, C, D, E, let us take the squares, 0, 1, 4, 9, 16, 25, and instead of the differences given above, the odd numbers, 1, 3, 5, 7, 9, will be disclosed; thus

$$\begin{array}{ccccccc} 0 & 1 & 4 & 9 & 15 & 25 \\ & 1 & 3 & 5 & 7 & 9 \end{array}$$

From which is evident that

$$1+3+5+7+9=25-0=25,$$
$$\text{and} \quad 3+5+7+9=25-1=24;$$

and the same will hold good whatever the number of terms or the differences may be, or whatever numbers are taken as the first and last terms.

Delighted by this easy and elegant theorem, our young friend considered a large number of numerical series, and also proceeded to the second differences or differences of the differences,[61] the third differences or the differences between the differences of the differences, and so on. He also observed that for the natural numbers, i. e., the numbers in order proceeding from 0, the second differences vanished, as also did the third differences for the squares, the fourth differences for the cubes, the fifth for the biquadrates, the sixth for the surdesolids,[62] and so on; also that the first differences for the natural numbers were constant and equal to 1; the second differences for the square, 1.2, or 2; the third for the cubes, 1.2.3, or 6; the fourth for the biquadrates, 1.2.3.4, or 24; the fifth for the surdesolids, 1.2.3.4.5, or 120, and so on. These things it is admitted had been previously noted by others,

but they were new to him, and by their easiness and elegance were in themselves an inducement to further advances. But especially he considered what he called "combinatory numbers," such as are usually tabulated as in the margin. Here a preceding series, either horizontal or vertical, always contains the first differences of the series immediately following it, the second differences of the one next after that, the third differences of the third, and so on. Also, each series, either horizontal or vertical contains the sums of the series immediately preceding it, the sums of the sums or the second sums of the series next before that, the third sums of the third, and so on. But, to give something not yet common knowledge, he also brought to light certain general theorems on differences and sums, such as the following. In the series, *a, b, c, d, e,* etc., where the terms continually decrease without limit we have

1	1	1	1	1	1	
1	2	3	4	5	6	
1	3	6	10	15	21	
1	4	10	20	35	56	
1	5	15	35	70	126	
1	6	21	56	126	252	
1	7	28	84	210	462	

Terms	a	b	c	d	e	etc.
1st diff.	f	g	h	i	k	etc.
2nd diff.	l	m	n	o	p	etc.
3rd diff.	q	r	s	t	u	etc.
4th diff.	β	γ	δ	ϵ	θ	etc.
etc.	γ	μ	ν	ρ	υ	etc.

Taking a as the first term, and ω as the last, he found

$$a - \omega = 1f + 1g + 1h + 1i + 1k + \text{etc.}$$
$$a - \omega = 1l + 2m + 3n + 4o + 5p + \text{etc.}$$
$$a - \omega = 1q + 3r + 6s + 10t + 15u + \text{etc.}$$
$$a - \omega = 1\beta + 4\gamma + 10\delta + 20\epsilon + 35\theta + \text{etc.}$$
$$\text{etc.}$$

Again we have[63]

$$a - \omega = \begin{cases} +1f \\ +1f - 1l \\ +1f - 2l + 1q \\ +1f - 3l + 3q - 1\beta \\ +1f - 4l + 6q - 4\beta + 1\lambda \\ \qquad \text{etc.} \quad \text{etc.} \quad \text{etc,} \end{cases}$$

Hence, adopting a notation invented by him at a later date, and denoting any term of the series generally by y (in which case $a = y$ as well), we may call the first difference dy, the second ddy, the third d^3y, the fourth d^4y; and calling any term of another of the series x, we may denote the sum of its terms by $\int x$, the sum of their sums or their second sum by $\int\int x$, the third sum by $\int^3 x$, and the fourth sum by $\int^4 x$. Hence, supposing that

$$1+1+1+1+1+\text{etc.} = x,$$

or that x represents the natural numbers, for which $dx = 1$, then

$$\begin{aligned} 1+3+\ 6+10+\text{etc.} &= \int x, \\ 1+4+10+20+\text{etc.} &= \int\int x, \\ 1+5+15+35+\text{etc.} &= \int^3 x, \end{aligned}$$

and so on. Finally it follows that

$$y - \omega = dy.x - ddy.\int x + d^3y.\int\int x - d^4y.\int^3 x + \text{etc.};$$

and this is equal to y, if we suppose that the series is continued to infinity, or that ω becomes zero. Hence also follows the sum of the series itself, and we have

$$\int y = yx - dy.\int x + ddy.\int\int x - d^3y.\int^3 x + \text{etc.}^{64}$$

These two like theorems possess the uncommon property that they are equally true in either differential calculus, the numerical or the infinitesimal; of the distinction between them we will speak later.[65]

However, the application of numerical truths to geometry, as well as the consideration of infinite series, was at that time at all events unknown to our young friend, and he was content with the satisfaction of having observed such things in series of numbers. Nor did he then, except for the most ordinary practical rules, know anything about geometry;[66] he had scarcely even considered Euclid with anything like proper attention, being fully occupied with other studies. However, by chance he came across the delightful contemplation of curves by Leotaud, in which the author deals with the quadrature of lunules, and Cavalieri's geometry of indivisibles;[67] having given these some slight consideration, he was delighted

with the facility of their methods. However, at that time he was in no mind to go fully in these more profound parts of mathematics, although just afterwards he gave attention to the study of physics and practical mechanics, as may be understood from his essay that he published on the *Hypothesis of Physics*.[68]

He then became a member of the Revision Council[69] of the Most Noble the Elector of Mainz; later, having obtained permission from this Most Gracious and Puissant Prince (for he had taken our young friend into his personal service when he was about to leave[70] and go further afield) to continue his travels, he set out for Paris in the year 1672. There he became acquainted with that genius Christian Huygens, to whose example and precepts he always declared that he owed his introduction to higher mathematics. At that time it so happened that Huygens was engaged on his work with regard to the pendulum. When Huygens brought our young friend a copy of this work as a present and in the course of conversation discussed the nature of the center of gravity, which our young friend did not know very much about, the former explained to him shortly what sort of thing it was and how it could be investigated.[71] This roused our young friend from his lethargy, for he looked upon it as something of a disgrace that he should be ignorant of such matters.[72]

Now it was impossible for him to find time for such studies just then; for almost immediately, at the close of the year, he crossed the Channel to England in the suite of the envoy from Mainz, and stayed there for a few weeks with the envoy. Having been introduced by Henry Oldenburg, at that time secretary to the Royal Society, he was elected a member of that illustrious body. He did not however at that time discuss geometry with any one (in truth at that time he was quite one of the common herd as regards this subject); he did not on the other hand neglect chemistry, consulting that excellent man Robert Boyle on several occasions. He also came across Pell accidentally, and he described to him certain of his own observations on numbers; and Pell told him that they were not new, but that it had been recently made known by Nicolaus Mercator, in his *Hyperbolae Quadratura*, that the differences of the powers of the natural numbers, when taken continuously, finally vanished; this made Leibniz obtain the work of Nicolaus Mercator.[73] At that time he did not become acquainted with Collins; and, although he conversed with Oldenburg on literary

matters, on physics and mechanics, he did not exchange with him even one little word on higher geomery, much less on the series of Newton. Indeed, that he was almost a stranger to these subjects, except perhaps in the properties of numbers, even that he had not paid very much attention to them, is shown well enough by the letters which he exchanged with Oldenburg, which have been lately published by his opponents. The same fact will appear clearly from those which they say have been preserved in England; but they suppressed them,[74] I firmly believe, because it would be quite clear from them that up to then there had been no correspondence between him and Oldenburg on matters geometrical. Nevertheless, they would have it credited (not indeed with the slightest evidence brought forward in favor of the supposition) that certain results obtained by Collins, Gregory and Newton, which were in the possession of Oldenburg, were communicated by him to Leibniz.

On his return from England to France in the year 1673,[75] having meanwhile satisfactorily performed his work for the Most Noble Elector of Mainz, he still by his favor remained in the service of Mainz; but his time being left more free, at the instigation of Huygens he began to work at Cartesian analysis (which aforetime had been beyond him),[76] and in order to obtain an insight into the geometry of quadratures he consulted the *Synopsis Geometriae* of Honoratus Fabri, Gregory St. Vincent, and a little book by Dettonville (i. e., Pascal).[77] Later on from one example given by Dettonville, a light suddenly burst upon him, which strange to say Pascal himself had not perceived in it. For when he proves the theorem of Archimedes for measuring the surface of a sphere or parts of it, he used a method in which the whole surface of the solid formed by a rotation round any axis can be reduced to an equivalent plane figure. From it our young friend made out for himself the following general theorem.[78]

Portions of a straight line normal to a curve, intercepted between the curve and an axis, when taken in order and applied at right angles to the axis give rise to a figure equivalent to the moment of the curve about the axis.[79]

When he showed this to Huygens the latter praised him highly and confessed to him that by the help of this very theorem he had found the surface of parabolic conoids and others of the same sort, stated without proof many years before in his work on the pendulum clock. Our young friend, stimulated by this and pondering

on the fertility of this point of view, since previously he had considered infinitely small things such as the intervals between the ordinates in the method of Cavalieri and such only, studied the triangle $_1Y\,D\,_2Y$, which he called the Characteristic Triangle,[80] whose sides $D\,_1Y$, $D\,_2Y$ are respectively equal to $_1X\,_2X$, $_1Z\,_2Z$,[81] parts of the coordinates or coabscissae AX, AZ, and its third side $_1Y\,_2Y$ a part of the tangent TV, produced if necessary.

Even though this triangle is indefinite (being infinitely small), yet he perceived that it was always possible to find definite triangles similar to it. For, suppose that AXX, AZZ are two straight lines at right angles, and AX, AZ the coabscissae, YX, YZ the coordi-

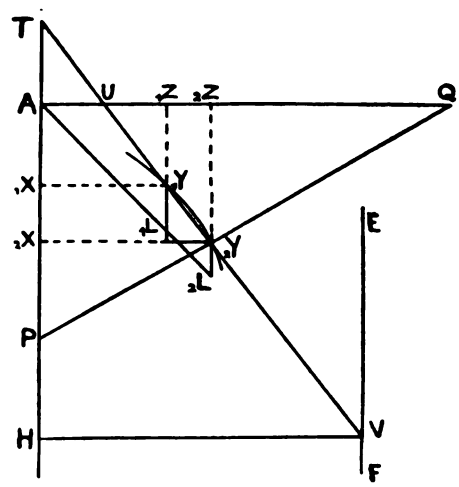


Fig. 3.

nates, TUV the tangent, PYQ the perpendicular, XT, ZU the subtangents, XP, ZQ the subnormals; and lastly let EF be drawn parallel to the axis AX; let the tangent TY meet EF in V, and from V draw VH perpendicular to the axis. Then the triangles $_1YD\,_2Y$, TXY, YZU, TAU, YXP, QZY, QAP, THV, and as many more of the sort as you like, are all similar. For example, from the similar triangles $_1YD\,_2Y$, $_2Y\,_2XP$, we have $P\,_2Y.\,_1YD = {}_2Y\,_2X.\,_2Y\,_1Y$; that is, the rectangle contained by the $P\,_2Y$ and $_1YD$ (or the element of the axis, $_1X\,_2X$) is equal to the rectangle contained by the ordinate $_2Y\,_2X$ and the element of the curve, $_1Y\,_2Y$, that is, to the moment of the element of the curve about the axis. Hence the

whole moment of the curve is obtained by forming the sum of these perpendiculars to the axis.

Also, on account of the similar triangles ${}_1YD\,{}_2Y$, THV, we have ${}_1Y\,{}_2Y : {}_2YD = TV = VH$, or $VH.{}_1Y\,{}_2Y = TV.{}_2YD$; that is, the rectangle contained by the constant length VH and the element of the curve, ${}_1Y\,{}_2Y$, is equal to the rectangle contained by TV and ${}_2YD$, or the element of the coabscissa, ${}_1Z\,{}_2Z$. Hence the plane figure produced by applying the lines TV in order at right angles to AZ is equal to the rectangle contained by the curve when straightened out and the constant length HV.

Again, from the similar triangles ${}_1YD\,{}_2Y$, ${}_2Y\,{}_2XP$, we have ${}_1YD : D\,{}_2Y = {}_2Y\,{}_2X : {}_2XP$, and thus ${}_2XP.{}_1YD = {}_2Y\,{}_2X.D\,{}_2Y$, or the sum of the subnormals ${}_2XP$, taken in order and applied to the axis, either to ${}_1YD$ or to ${}_1X\,{}_2X$ and their elements ${}_2YD$, taken in order. But straight lines that continually increase from zero, when each is multiplied by its element of increase, form altogether a triangle. Let then AZ always be equal to ZL, then we get the right-angled triangle AZL, which is half the square on AZ; and thus the figure that is produced by taking the subnormals in order and applying them perpendicular to the axis will be always equal to half the square on the ordinate. Thus, to find the area of a given figure, another figure is sought such that its subnormals are respectively equal to the ordinates of the given figure, and then this second figure is the quadratrix of the given one; and thus from this extremely elegant consideration we obtain the reduction of the areas of surfaces described by rotation[82] to plane quadratures, as well as the rectification of curves; at the same time we can reduce these quadratures of figures to an inverse problem of tangents. From these results,[83] our young friend wrote down a large collection of theorems (among which in truth there were many that were lacking in elegance) of two kinds. For in some of them only definite magnitudes were dealt with, after the manner not only of Cavalieri, Fermat, Honoratus Fabri, but also of Gregory St. Vincent, Guldinus, and Dettonville; others truly depended on infinitely small magnitudes, and advanced to a much greater extent. But later our young friend did not not trouble to go on with these matters, when he noticed that the same method can be brought into use and perfected by not only Huygens, Wallis, Van Huraet, and Neil, but also by James Gregory and Barrow. However it did not seem to me to be altogether useless to explain at this junc-

ture, as is plain from what I have given,[84] the steps by which he attained to greater things, and also the manner in which, as if led by the hand, those who are at present but beginners[85] with regard to the more abstruse parts of geometry may hope to rise to greater heights.

Now Leibniz worked these things out at Paris in the year 1673 and part of 1674. But in the year 1674 (so much it is possible to state definitely), he came upon the well-known arithmetical teragonism;[86] and it will be worth while to explain how this was accomplished. He once happened to have occasion to break up an area into triangles formed by a number of straight lines meeting in a

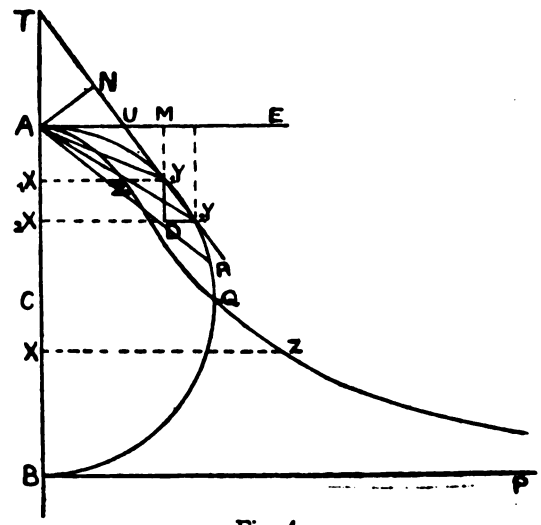


Fig. 4.

point, and he perceived that something new could be readily obtained from it.[87]

In Fig. 4, let any number of straight lines, AY, be drawn to the curve AYR, and let any axis AC be drawn, and AE a normal or coaxis to it; and let the tangent at Y to the curve cut them in T and U. From A draw AN perpendicular to the tangent; then it is plain that the elementary triangle $A\,{}_1Y\,{}_2Y$ is equal to half the rectangle contained by the element of the curve ${}_1Y\,{}_2Y$ and AN. Now draw the characteristic triangle mentioned above, ${}_1YD\,{}_2Y$, of which the hypotenuse is a portion of the tangent or the element of the arc, and the sides are parallel to the axis and the coaxis. It

is then plain from the similar triangles ANU, ${}_1YD_2Y$, that ${}_1Y_2Y : {}_1YD = AU : AN$, or $AU . {}_1YD$ or $AU . {}_1X_2X$ is equal to $AN . {}_1Y_2Y$, and this, as has been already shown, is equal to double the triangle A_1Y_2Y. Thus if every AU is supposed to be transferred to XY, and taken in it as AZ,[88] then the trilinear space AXZA so formed will be equal to twice the segment AYA,[89] included between the straight line AY and the arc AY. In this way are obtained what he called the figures of segments or the proportionals of a segment. A similar method holds good for the case in which the point is not taken on the curve, and in this manner he obtained the proportional trilinear figures for sectors cut off by lines meeting in the point; and even when the straight lines had their extremities not in a line but in a curve (which one after the other they touched), none the less on that account were useful theorems made out.[90] But this is not a fit occasion to follow out such matters; it is sufficient for our purpose to consider the figures of segments, and that too only for the circle. In this case, if the point A is taken at the beginning of the quadrant AYQ, the curve AZQZ will cut the circle at Q, the other end of the quadrant, and thence descending will be asymptotic to the base BP (drawn at right angles to the diameter at its other end B); and, although extending to infinity, the whole figure, included between the diameter AB, the base BP...., and the curve AZQZ.... asymptotic to it, will be equal to the circle on AB as diameter.

But to come to the matter under discussion, take the radius as unity, put AX or UZ = x, and AU or AZ = z, then we have $x = 2z : 1 + zz$;[91] and the sum of all the x's applied to AU, which at the present time we call $\int x\,dz$, is the trilinear figure AUZA, which is the complement of the trilinear figure AXZA, and this has been shown to be double the circular segment.

The author obtained the same result by the method of transmutations, of which he sent an account to England.[92] It is required to form the sum of all the ordinates $\sqrt{(1 - xx)} = y$; suppose $y = \pm 1 \mp xz$, from which $x = 2z : 1 + zz$, and $y = \pm zz \mp 1 : zz + 1$; and thus again all that remains to be done is the summation of rationals.

This seemed to him to be a new and elegant method, as it did to Newton also, but it must be acknowledged that it is not of universal application. Moreover it is evident that in this way the arc may be obtained from the sine, and other things of

the same kind, but indirectly. So when later he heard that these things had been derived in a direct manner by Newton with the help of root-extractions,[93] he was desirous of getting a knowledge of the matter.

From the above it was at once apparent that, using the method by which Nicolaus Mercator had given the arithmetical tetragonism of the hyperbola by means of an infinite series, that of the circle might also be given, though not so symmetrically, by dividing by $1+zz$, in the same way that the former had divided by $1+z$. The author, however, soon found a general theorem for the area of any central conic. Namely, the sector included by the arc of a conic section, starting from the vertex, and two straight lines joining its ends to the center, is equal to the rectangle contained by the semi-transverse axis and a straight line of length

$$t \pm \frac{1}{3}t^3 + \frac{1}{5}t^5 \pm \frac{1}{7}t^7 + \ldots\ldots,$$ [94]

where t is the portion of the tangent at the vertex intercepted between the vertex and the tangent at the other extremity of the arc, and unity is the square on the semi-conjugate axis or the rectangle contained by the halves of the latus-rectum and the transverse axis, and $\pm$ is to be taken to mean + for the hyperbola and – for the circle or the ellipse. Hence if the square of the diameter is taken to be unity, then the area of the circle is

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}$$

When our friend showed this to Huygens, together with a proof of it, the latter praised it very highly, and when he returned the dissertation said, in the letter that accompanied it, that it would be a discovery always to be remembered among mathematicians, and that in it the hope was born that at some time it might be possible that the general solution should be obtained either by exhibiting its true value or by proving the impossibility of expressing it in recognized numbers.[95] There is no doubt that neither he nor the discoverer, nor yet any one else in Paris, had heard anything at all by report concerning the expression of the area of a circle by means of an infinite series of rationals (such as afterward it became known had been worked out by Newton and

Gregory). Certainly Huygens did not, as is evident from the short letter from him that I give herewith.[96]... Thus Huygens believed that it was now proved for the first time that the area of a circle was exactly equal to a series of rational quantities. Leibniz (relying on the opinion of Huygens, who was well versed in such matters), believed the same thing and so wrote those two letters to Oldenburg in 1674, which his opponents have published, in which he announces it as a new discovery;[97] indeed he went so far as to say that he, before all others, had found the magnitude of the circle expressed as a series of rational numbers, as had already been done in the case of the hyperbola.[98] Now, if Oldenburg had already communicated to him during his stay in London the series of Newton and Gregory,[99] it would have been the height of impudence for him to have dared to write in this way to Oldenburg; and either forgetfulness or collusion on the part of Oldenburg in not charging him with the deceit. For these opponents publish the reply of Oldenburg, in which he merely points out (he says "I do not wish you to be unaware....") that similar series had been noted by Gregory and Newton; and these things also he communicated in the year following in a letter (which they publish) written in the month of April.[100] From which it can be seen that they are blinded with envy or shameless with spite who dare to pretend that Oldenburg had already communicated those things to him in the preceding year. Yet there may be some blindness in their spite, because they do not see that they publish things by which their lying statements are refuted, nor that it would have been far better to have suppressed these letters between him and Oldenburg, as they have done in the case of others, either wholly or in part. Besides, from this time onwards he begins to correspond with Oldenburg about geometry; that is, from the time when he, who up till then had been but a beginner in this subject, first found out anything that he considered worthy to be communicated; and former letters written from Paris on March 30, April 26, May 24, and June 8, in the year 1673, which they say they have at hand but suppress, together with the replies of Oldenburg, must undoubtedly have dealt with other matters and have nothing in them to render those fictitious communications from Oldenburg the more deserving of belief. Again, when our young friend heard that Newton and Gregory had discovered their series by the extraction of roots,[101] he acknowledged that this was new to him, nor at first did he understand it very

much; and he confessed as much quite frankly and asked for information on certain points, especially for the case in which reciprocal series were sought, by means of which from one infinite series the root was extracted by means of another infinite series. And from this also it is evident that what his opponents assert, that Oldenburg communicated the writings of Newton to him, is false; for if that were the truth, there would have been no need to ask for further information. On the other hand, when he began to develop his differential calculus, he was convinced that the new method was much more universal for finding infinite series without root-extractions, and adapted not only for ordinary quantities but for transcendent quantities as well, by assuming that the series required was given; and he used this method to complete his short essay on the arithmetical quadrature; in this he also included other series that he had discovered, such as an expression for the arc in terms of the sine or the complement of the sine, and conversely he showed how, by this same method, to find the sine or cosine when the arc was given.[102] This too is the reason why later he stood in no need of other methods than his own; and finally, he published his own new way of obtaining series in the *Acta Eruditorum*. Moreover, as it was at this time, just after he had published the essay on the Arithmetical Quadrature in Paris, that he was recalled to Germany, having perfected the technique of the new calculus he paid less attention to the former methods.

Now it is to be shown how, little by little, our friend arrived at the new kind of notation that he called the differential calculus. In the year 1672, while conversing with Huygens on the properties of numbers, the latter propounded to him this problem:[103]

To find the sum of a decreasing series of fractions, of which the numerators are all unity and the denominators are the triangular numbers; of which he said that he had found the sum among the contributions of Hudde on the estimation of probability. Leibniz found the sum to be 2, which agreed with that given by Huygens. While doing this he found the sums of a number of arithmetical series of the same kind in which the numbers are any combinatory numbers whatever, and communicated the results to Oldenburg in February 1673, as his opponents have stated. When later he saw the Arithmetical Triangle of Pascal, he formed on the same plan his own Harmonic Triangle.

Arithmetical Triangle

in which the fundamental series is an arithmetical progression

1, 2, 3, 4, 5, 6, 7, . . .

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1
1	7		21		35		35		21		7	1

Harmonic Triangle[104]

in which the fundamental series is a harmonical progression

						$\frac{1}{1}$						
					$\frac{1}{2}$		$\frac{1}{2}$					
				$\frac{1}{3}$		$\frac{1}{6}$		$\frac{1}{3}$				
			$\frac{1}{4}$		$\frac{1}{12}$		$\frac{1}{12}$		$\frac{1}{4}$			
		$\frac{1}{5}$		$\frac{1}{20}$		$\frac{1}{30}$		$\frac{1}{20}$		$\frac{1}{5}$		
	$\frac{1}{6}$		$\frac{1}{30}$		$\frac{1}{60}$		$\frac{1}{60}$		$\frac{1}{30}$		$\frac{1}{6}$	
$\frac{1}{7}$		$\frac{1}{42}$		$\frac{1}{105}$		$\frac{1}{140}$		$\frac{1}{105}$		$\frac{1}{42}$		$\frac{1}{7}$

where, if the denominators of any series descending obliquely to infinity or of any parallel finite series, are each divided by the term that corresponds in the first series, the combinatory numbers are produced, namely those that are contained in the arithmetical triangle. Moreover this property is common to either triangle, namely, that the oblique series are the sum- and difference-series of one another. In the Arithmetical Triangle any given series is the sum-series of the series that immediately precedes it, and the difference-

series of the one that follows it; in the Harmonic Triangle, on the other hand, each series is the sum-series of the series following it, and the difference-series of the series that precedes it. From which it follows that

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\text{etc.}=\frac{1}{0}$$

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{28}+\text{etc.}=\frac{2}{1}$$

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}+\frac{1}{56}+\frac{1}{84}+\text{etc.}=\frac{3}{2}$$

$$\frac{1}{1}+\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}+\frac{1}{126}+\frac{1}{210}+\text{etc.}=\frac{4}{3}$$

and so on.

Now he had found out these things before he had turned to Cartesian analysis; but when he had had his thoughts directed to this, he considered that any term of a series could in most cases be denoted by some general notation, by which it might be referred to some simple series. For instance, if the general term of the series of natural numbers is denoted by x, then the general term of the series of squares would be x^2, that of the cubes would be x^3, and so on. Any triangular number, such as 0, 1, 3, 6, 10, would be

$$\frac{x.x+1}{1.2} \text{ or } \frac{xx+x}{2},$$

any pyramidal number, such as 0, 1, 4, 10, 20, etc., would be

$$\frac{x.x+1.x+2}{1.2.3} \text{ or } \frac{x^3+3xx+2x}{6},$$

and so on.

From this it was possible to obtain the difference-series of a given series, and in some cases its sum as well, when it was expressed numerically. For instance, the square is xx, the next greater square is $xx+2x+1$, and the difference of these is $2x+1$; i. e., the series of odd numbers is the difference-series for the series of squares. For, if x is 0, 1, 2, 3, 4, etc., then $2x+1$ is 1, 3, 5, 7, 9. In the same way the difference between x^3 and $x^3+3xx+3x+1$ is $3xx+3x+1$, and thus the latter is the general term of the difference-series

for the series of cubes. Further, if the value of the general term can thus be expressed by means of a variable x so that the variable does not enter into a denominator or an exponent, he perceived that he could always find the sum-series of the given series. For instance, to find the sum of the squares, since it is plain that the variable cannot be raised to a higher degree than the cube, he supposed its general term z to be

$$z = lx^3 + mxx + nx, \text{ where } dz \text{ has to be } xx;$$

we have $dz = l\,d(x^3) + m\,d(xx) + n$, (where dx is taken $= 1$); now $d(x^3) = 3xx + 3x + 1$, and $d(xx) = 2x + 1$, as already found; hence

$$dz = 3lxx + 3lx + l + 2mx + m + n \;\text{ ☌ }\; xx,^{105}$$

therefore $l = \frac{1}{3}$, $m = -\frac{1}{2}$, and $\frac{1}{3} - \frac{1}{2} + n = 0$, or $n = \frac{1}{6}$;

and the general term of the sum-series for the squares is

$$\frac{1}{3}x^3 - \frac{1}{2}xx + \frac{1}{6}x \text{ or } 2x^3 - 3xx + x, :6.^{106}$$

As an example, if it is desired to find the sum of the first nine or ten squares, i. e., from 1 to 81 or from 1 to 100, take for x the values 10 or 11, the numbers next greater than the root of the last square, and $2x^3 - 3xx + x, :6$ will be $2000 - 300 + 10, :6 = 285$, or $2.1331 - 3.121 + 11, :6 = 385$. Nor is it much more difficult with this formula to sum the first 100 or 1000 squares. The same method holds good for any powers of the natural numbers or for expressions which are made up from such powers, so that it is always possible to sum as many terms as we please of such series by a formula. But our friend saw that it was not always easy to proceed in the same way when the variable entered into the denominator, as it was always possible to find the sum of a numerical series; however, on following up this same analytical method, he found in general, and published the result in the *Acta Eruditorum*, that a sum-series could always be found, or the matter be reduced to finding the sum of a number of fractional terms such as $1/x$, $1/xx$, $1/x^3$, etc, which at any rate, if the number of terms taken is finite, can be summed, though hardly in a short way (as by a formula); but if it is a question of an infinite number of terms, then terms such as $1/x$ cannot be summed at all, because the total of an infinite number of terms of such a series is an infinite quantity,

but that of an infinite number of terms such as $1/xx$, $1/x^3$, etc., make a finite quantity, which nevertheless could not up till now be summed, except by taking quadratures. So, in the year 1682, in the month of February, he noted in the *Acta Eruditorum* that if the numbers 1.3, 3.5, 5.7, 7.9, 9.11, etc., or 3, 15, 35, 63, 99, etc., are taken, and from them is formed the series of fractions

$$\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\frac{1}{99}+\text{etc.},$$

then the sum of this series continued to infinity is nothing else but ½; while, if every other fraction is left out, $\frac{1}{3}+\frac{1}{35}+\frac{1}{99}+$ etc. expresses the magnitude of a semicircle of which the square on the diameter is represented by 1.[107]

Thus, suppose $x = 1, 2, 3,$ etc.[108] Then the general term of

$$\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\frac{1}{63}+\text{etc. is } \frac{1}{4xx+8x+3};$$

it is required to find the general term of the sum-series.

Let us try whether it can have the form $e/(bx+c)$, the reasoning being very simple; then we shall have

$$\frac{e}{bx+c}-\frac{e}{bx+b+c}=\frac{eb}{bbxx+bbx+bc+2bcx+cc}\text{ or }\frac{1}{4xx+8x+3};$$

hence, equating coefficients in these two formulas, we have

$$b = 2,\ eb = 1,\text{ or } e = \tfrac{1}{2},$$

$$bb+2bc = 8,\text{ or } 4+4c = 8,\text{ or } c = 1;$$

and finally we should have also $bc+cc = 3$, which is the case. Hence the general term of the sum-series is $(1:2)/(2x+1)$ or $1/(4x+2)$, and these numbers of the form $4x+2$ are the doubles of the odd numbers. Finally he gave a method for applying the differential calculus to numerical series when the variable entered into the exponent, as in a geometrical progression, where, taking any radix b the term is b^x, where x stands for a natural number. The terms of the differential series will be $b^{x+1}-b^x$, or $b^x(b-1)$; and from this it is plain that the differential series of the given geometrical series is also a geometrical series proportional to the given series. Thus the sum of a geometrical series may be obtained.

But our young friend quickly observed that the differential calculus could be employed with diagrams in an even more wonderfully simple manner than it was with numbers, because with diagrams the differences were not comparable with the things which differed; and as often as they were connected together by addition or subtraction, being incomparable with one another, the less vanished in comparison with the greater; and thus irrationals could be differentiated no less easily than surds, and also, by the aid of logarithms, so could exponentials. Moreover, he observed that the infinitely small lines occurring in diagrams were nothing else but the momentaneous differences of the variable lines. Also, in the same way as quantities hitherto considered by analytical mathematicians had their functions such as powers and roots, so also such quantities as were variable had new functions, namely, differences. Also, that as hitherto we had x, xx, x^3, etc., y, yy, y^3, etc., so now it was possible to have dx, ddx, d^3x, etc., dy, ddy, d^3y, and so forth. In the same way, that it was possible to express curves, which Descartes had excluded as being "mechanical," by equations of position, and to apply the calculus to them and thus to free the mind from a perpetual reference to diagrams. In the applications of the differential calculus to geometry, differentiations of the first degree were equivalent to nothing else but the finding of tangents, differentiations of the second degree to the finding of osculating circles (the use of which was introduced by our friend); and that it was possible to go on in the same fashion. Nor were these things only of service for tangents and quadratures, but for all kinds of problems and theorems in whch the differences were intermingled with integral terms (as that brilliant mathematician Bernoulli called them), such as are used in physico-mechanical problems.

Thus it follows generally that if any series of numbers or lines of a figure have a property that depends on two, three or more consecutive terms, it can be expressed by an equation involving differences of the first, second, third, or higher degree. Moreover, he discovered general theorems for any degree of the differences, just as we have had theorems of any degree, and he made out the remarkable analogy between powers and differences published in the *Miscellania Berolinensia*.

If his rival had known of these matters, he would not have used dots to denote the degrees of the differences,[109] which are useless for expressing the general degree of the differences, but

would have used the symbol d given by our friend or something similar, for then d^e can express the degree of the difference in general. Besides everything which was once referred to figures, can now be expressed by the calculus.

For $\sqrt{(dxdx+dydy)}$[110] was the element of the arc of a curve, ydx was the element of its area; and from that it is immediately evident that $\int y\,dx$ and $\int x\,dy$ are the complements of one another, since $d(xy) = x\,dy + y\,dx$, or conversely, $xy = \int x\,dy + \int y\,dx$, however these figures vary from time to time; and from this, since $xyz = \int xy\,dz + \int xz\,dy + \int yz\,dx$, three solids are also given that are complementary, every two to the third. Nor is there any need for him to have known those theorems which we deduced above from the characteristic triangle; for example, the moment of a curve about the axis is sufficiently expressed by $\int x\sqrt{(dxdx+dydy)}$. Also what Gregory St. Vincent has concerning *ductus*, what he or

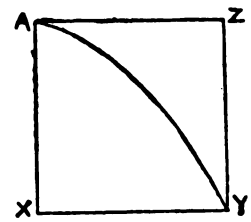


Fig. 5.

Pascal had concerning *ungulae* and *cunei*,[111] every one of these is immediately deduced from a calculus such as this. Thus Leibniz saw with delight those discoveries that he had applauded in others obtained by himself, and thereupon he left off studying them at all closely, because all of them were contained in a calculus such as his.

For example, the moment of the figure AXYA (Fig. 5) about the axis is $\frac{1}{2}\int yy\,dx$, the moment of the figure about the tangent at the vertex is $\int xy\,dx$, the moment of the complementary trilinear figure AZYA about the tangent at the vertex is $\frac{1}{2}\int xx\,dy$. Now these two last moments taken together yield the moment of the circumscribed rectangle AXYZ about the tangent at the vertex, and are complementary to one another.

However, the calculus also shows this without reference to any figure, for $\frac{1}{2}d(xxy) = xy\,dx + \frac{1}{2}xx\,dy$; so that now there is need

for no greater number of the fine theorems of celebrated men for Archimedean geometry, than at most those given by Euclid in his Book II or elsewhere, for ordinary geometry.

It was good to find that thereafter the calculus of transcendent quantities should reduce to ordinary quantities, and Huygens was especially pleased with this. Thus, if it is found that

$$2\int \frac{dy}{y} = 3\int \frac{dx}{x},$$

then from this we get $yy = x^3$, and this too from the nature of logarithms combined with the differential calculus, the former also being derived from the same calculus. For let $x^m = y$, then $mx^{n-1}\,dx = dy$. Hence, dividing each side by equal things, we have

$$m\frac{dx}{x} = \frac{dy}{y}.$$

Again, from the equation, $m \log x = \log y$, we have

$$\log x : \log y = \int \frac{dx}{x} : \int \frac{dy}{y}.$$ [113]

By this the exponential calculus is rendered practicable as well. For let $y^x = z$, then $x \log y = \log z$, $dx \log y + x\,dy : y = dz : z$.

In this way we free the exponents from the variable, or at other times we may transpose the variable exponent with advantage under the circumstances. Lastly, those things that were once held in high esteem are thus made a mere child's-play.

Now of all this calculus not the slightest trace existed in all the writings of his rival before the principles of the calculus were published by our friend;[114] nor indeed anything at all that Huygens or Barrow had not accomplished in the same way, in the cases where they dealt with the same problems.

But how great was the extent of the assistance afforded by the use of this calculus was candidly acknowledged by Huygens; and this his opponents suppress as much as ever they can, and straightway go on with other matters, not mentioning the real differential calculus in the whole of their report. Instead, they adhere to a large extent to infinite series, the method for which no one denies that his rival brought out in advance of all others. For those things which he said enigmatically, and explained at a much

later date, are all they talk about, namely, fluxions and fluents, i. e., finite quantities and their infinitely small elements; but as to how one can be derived from the other they offer not the slightest suggestion. Moreover, while he considers nascent or evanescent ratios, leading straight away from the differential calculus to the method of exhaustions, which is widely different from it (although it certainly also has its own uses), he proceeds not by means of the infinitely small, but by ordinary quantities, though these latter do finally become the former.

Since therefore his opponents, neither from the *Commercium Epistolicum* that they have published, nor from any other source, brought forward the slightest bit of evidence whereby it might be established that his rival used the differential calculus before it was published by our friend; therefore all the accusations that were brought against him by these persons may be treated with contempt as beside the question. They have used the dodge of the petti-fogging advocate[115] to divert the attention of the judges from the matter on trial to other things, namely to infinite series. But even in these they could bring forward nothing that could impugn the honesty of our friend, for he plainly acknowledged the manner in which he had made progress in them; and in truth in these also, he finally attained to something higher and more general.

SUPPLEMENT.

Barrow, *Lectiones Geometricae*, Lect. XII, Prop. 1, 2, 3.

[Page 105, First Edition, 1670.]

General foreword. We will now proceed with the matter in hand; and, in order that we may save time and words, it is to be observed everywhere in what now follows that AB is some curved line, such as we shall draw, of which the axis is AD; to this axis all the straight lines BD, CA, MF, NG are applied perpendicular; the arc MN is indefinitely small; the straight line $\alpha\beta =$ arc AB, the straight line $\alpha\mu =$ arc AM, and $\mu\nu =$ arc MN; also lines applied to $\alpha\beta$ are perpendicular to it. On this understanding:

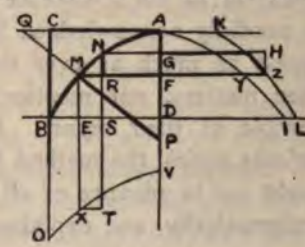


Fig. 6.

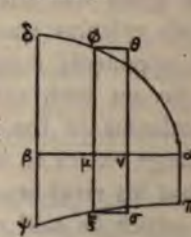


Fig. 7.

1. Let MP be perpendicular to the curve AB, and the lines KZL, $\alpha\phi\delta$ such that FZ = MP, $\mu\phi$ = MF. Then the spaces $\alpha\beta\delta$, ADLK are equal.

For the triangles MRN, PFM *are similar,* MN : NR = PM : MF,

$$MN.MF = NR.PM;$$

that is, on substituting the equal quantities,

$$\mu\nu.\mu\phi = FG.FZ, \text{ or rect. } \mu\theta = \text{rect. } FH.$$

But the space $\alpha\beta\delta$ only differs in the slightest degree from an infinite number of rectangles such as $\mu\theta$, and the space ADLK is equivalent to an equal number of rectangles such as FH. Hence the proposition follows.

2. Hence, *if the curve* AMB *is rotated about the axis* AD, *the ratio of the surface produced to the space* ADLK *is that of the circumference of a circle to its diameter*; whence if the space ADLK is known, the said surface is known.

Some time ago I assigned the reason why this was so.

3. Hence, *the surfaces of the sphere, both the spheroids, and the conoids receive measurement.* For if AD is the axis of the conic section, etc.

NOTES.

[1] For abbreviations used in this article for these and other publications, see the list on pp. 483-485.

[2] G. 1848, p. 29; see also G. *math.*, III, pp. 71, 72, and Cantor, III, p. 40.

[3] A fair-minded consideration, like everything emanating from the pen of De Morgan, is given of the matter in a recent edition of his *Essays on the Life and Work of Newton.* The tale is told with the charm characteristic of De Morgan, and the edition is rendered very valuable by the addition of notes, commentary, and a large number of references supplied by the editor, P. E. B. Jourdain (Open Court Publishing Co.). Special attention is directed to De Morgan's summary of the unfairness of the case in Note 3 at the foot of pages 27-28.

[4] See under 11 below: also cf. the original Latin as given in G. 1846, p. 4, *"per amicum conscium."*

[5] The account here given is substantially that given by Gerhardt in an article in Grunert's *Archiv der Mathematik und Physik,* 1856; pp. 125-132.

This article is written in contradiction to the view taken by Weissenborn in his *Principien der höheren Analysis,* Halle, 1856. It is worthy of remark that the partisanship of Gerhardt makes him omit in this article all mention of the review which Leibniz wrote for the *Acta Eruditorum* on Newton's work, *De Quadratura Curvarum,* which really drew upon him the renewal of the attack, by Keill. The passage which was objected to by the English mathematicians as being tantamount to a charge of plagiarism, in addition to the insult implied, according to their thinking, in making Newton the fourth proportional to Cavalieri, Fabri and Leibniz, is however given by Gerhardt in his preface to the *Historia* (G. 1846, p. vii). The following is a translation:

"Instead of the differences of Leibniz, Newton employs, and always has employed fluxions, which are as near as possible to augments of fluents—and these he has used both in his *Principia Naturæ Mathematica,* as well as in other works published later, with elegance; just as Honoratus Fabri in his *Synopsis Geometrica* has substituted increases of motions for the method of Cavalieri."

[6] Fatio's correspondence with Huygens is to be found in *Ch. Hugenii aliorumque seculi XVII virorum celebrium exercitationes mathematicae et philosophicae,* ed. Uylenbroeck, 1833.

[7] Bernoulli (Jakob), *Opera,* Vol. I, p. 431.

[8] *Ibid.,* p. 453.

[9] Cantor, III, p. 221.

[10] In the opening paragraph of the postscript, page 583.

[11] The account which follows is taken from Williamson's article, "Infinitesimal Calculus," in the *Times* edition of the *Encyc. Brit.* The memoir referred to contains a passage, of which the following is a translation (G., 1846, p. v):

"Perhaps the distinguished Leibniz may wish to know how I came to be acquainted with the calculus that I employ. I found out for myself its general principles and most of the rules in the year 1687, about April and the months following, and thereafter in other years; and at the time I thought that nobody besides myself employed that kind of calculus. Nor would I have known any the less of it, if Leibniz had not yet been born. And so let him be lauded by other disciples, for it is certain that I cannot do so. This will be all the more obvious, if ever the letters which have passed between the distinguished Huygens and myself come to be published. However, driven thereto by the very evidence of things, I am bound to acknowledge that Newton was the first, and by many years the first, inventor of this calculus; from whom, whether Leibniz, the second inventor, borrowed anything, I prefer that the decision should lie, not with me, but with others who have had sight of the paper of Newton, and other additions to this same manuscript. Nor does the silence of the more modest Newton, or the forward obtrusiveness of Leibniz. . . ."

Truly another Roland in the field, and one in a vicious mood. What with other claimants to the method, such as Slusius, etc., at least as far as the differentiation of implicit functions of two variables is concerned, it would almost seem that the infinitesimal calculus, like Topsy, "just growed."

[12] See De Morgan's *Newton*, p. 26 and pp. 148, 149, where the Scholium is translated. The original Latin of this Scholium to Lemma II of Book II of the *Principia*, the altered Scholium that appeared in the second and third editions, with a note remarking on the change, will be found on pp. 48, 49, in Book II of the "Jesuits' Edition' 'of Newton (Editio Nova, edited by J. M. F. Wright, Glasgow, 1822; the third and best edition of the work by Le Sœur and Jacquier).

[13] *Phil. Trans.*, 1708; see also Cantor, III, p. 299.

[14] For a discussion, see Rosenburger, *Isaac Newton und seine physikalischen Principien*, Leipsic, 1895.

[15] The manner of the opening of this postscript would seem to indicate that something had been mentioned with regard to the matter of his irritation about imputed obligations to Barrow in the body of the letter; this cannot be ascertained, for Gerhardt does not quote the letter in connection.

[16] Leibniz can hardly with justice call Barrow his contemporary; Barrow anticipated him by half a dozen years at least. For Barrow had published his *Lectiones Geometricae* in 1670, while the very earliest date at which Leibniz could have obtained his results is the end of 1672; and there is reason to believe, as I have shown in my edition of the *Lectiones*, that Barrow was in possession of his method many years before publication, and had most probably communicated his secret to Newton in 1664.

[17] It is to be noted that the sole topic of this postscript is geometry, of which Leibniz candidly states that he knew practically nothing in 1672.

[18] Most probably the *Institutiones arithmeticae* of Johann Lantz, published at Munich in 1616; Cantor, III, p. 40.

[19] Possibly the *Geometria practica* of Christopher Clavius, better known as an editor of Euclid; he was the professor at Rome under whom Gregory St. Vincent studied. There are repeated references to Clavius in Cantor, II and III, Index, *q. v.*

It is worth remarking that neither Lanzius nor Clavius are mentioned in the *Historia*.

[20] It has been stated that, according to Descartes's own words, the intricacies of his *Geométrie* were intentional; it certainly has the character of a challenge to his contemporaries. There is no preparation, such as marks a book of the present day on coordinate geometry; Descartes starts straightway on the solution of a problem given up as insoluble by the ancients. No wonder that young Leibniz found some difficulty with his first attempt to read it.

[21] In 1635, Cavalieri published his *Geometria indivisibilibus*, and thus laid the foundation stone of the integral calculus. It would seem that Roberval was really the first inventor, or at least an independent inventor of the method; but he lost credit for it because he did not publish it, preferring to keep it to himself for his own use. Other examples of this habit are common among the mathematicians of the time.

[22] The book referred to was published in 1654. It appeared as the second volume of a work whose first volume was a critique and refutation of the quadrature of the circle published by Gregory St. Vincent; this second volume was not the work of Leotaud, as the second part of the title showed: "necnon CURVILINEORUM CONTEMPLATIO, olim inita ab ARTUSIO DE LIONNE, Vapincensi Episc." It therefore appears to have been an edited reprint of the work of De Lionne, the bishop of Gap (ancient name, Vapincum). Since part of this treatise is devoted to the "lunules of Hippocrates" (see Cantor, I, pp. 192-194), it may have had some influence with Leibniz in giving him the first idea for his evaluation of π.

[23] Literally, "I was about to swim without corks."

[24] Leibniz here would appear to assert that he had considered some form of rectangular coordinate geometry, the association with the name of Descartes being fairly conclusive. Vieta's *In Artem Analyticam Isagoge*, explained how algebra could be applied to the solution of geometrical problems (Rouse Ball); for further information see Cantor.

[25] This seems to have been an improvement on the adding machine of Pascal, adapting it to multiplication, division and extraction of roots. Pascal's machine was produced in 1642, and Leibniz's in 1671.

[26] Huygens's *Horologium Oscillatorium* was published in 1673; we are thus provided with an exact date for the occurrence of the conversation that set Leibniz on to read Pascal and St. Vincent. This was after his first visit to London, from which he returned in March, "having utilized his stay in London to purchase a copy of Barrow's *Lectiones*, which Oldenburg had brought to his notice" (Zeuthen, *Geschichte der Mathematik im XVI. und XVII. Jahrhundert*; German edition by Mayer, p. 66). Leibniz himself mentions in a letter to Oldenburg, dated April 1673, that he has done so. Gerhardt (G. 1855, p. 48) states that he has seen, in the Royal Library of Hanover the copy of Barrow's *Lectiones Geometricae*, so that it must have been the combined edition of the Optics and the Geometry, published in 1670, that Leibniz bought.

Thus, before he is advised to study Pascal by Huygens, he has already in his possession a copy of Barrow. It is idle that any one should suppose that Leibniz bought this book on the recommendation of a friend in order merely to possess it; Leibniz bought books, or borrowed them, for the sole purpose of study. Unless we are to look upon this account of his reading as the result of lack of memory extending back for thirty years, there is only one conclusion to come to, barring of course the obviously brutal one that Leibniz lied; and this conclusion is that at the first reading the only thing that Leibniz could follow in Barrow was the part that he marked *Novi dudum* ("Knew this before"), and this was the appendix to Lecture XI, which dealt with the *Cyclometria* of Huygens, as Barrow calls the book entitled *De Circuli Magnitudine Inventa*. The absence of any more such remarks is almost proof positive that Leibniz knew none of the rest before. Hence he must have read the Barrow before he had filled those "hundreds of sheets" that he speaks of

later, with geometrical theorems that he has discovered; for at the end of the postscript we are considering he states that "in Barrow, *when his Lectures appeared*, I found the greater part of my theorems anticipated." There is something very wrong somewhere; for this would appear to state that it was the second edition of Barrow, published in 1874, that Leibniz had bought; it is impossible, as the words of Leibniz stand, that they should refer to the 1670 edition, for it had been published before Leibniz arrived in Paris. It is however certain from Leibniz's letter to Oldenburg that it could not be the 1674 edition, for the date of the letter is 1673.

In this letter Leibniz only makes a remark on the optical portion; but it could not have been the separate edition of the Optics, published in 1669, for Gerhardt states that the copy he has seen contains the Geometry with notes in the margin.

To those who have ever waded through the combined edition of Barrow's Optics and Geometry, it may be that rather a startling suggestion will occur. It was sheer ill-luck that drove Leibniz, after studying the Optics (perhaps on the journey back from London, for we know that this was a habit of his), to get tired of the five preliminary geometrical lectures in all their dryness, and on reaching home, just to skim over the really important chapters, *missing all the important points*, and just the name of Huygens catching his eye. This is a new suggestion as far as I am aware; everybody seems to decide between one of two things, either that Leibniz never read the book until the date he himself gives, "*Anno Domini* 1675 as far as I remember," or else that he purposely lied. I will return to this point later; meanwhile see Cantor, III, pp. 161-163, and consult the references given in the footnotes to these pages; the pros and cons of the conflict between probability and Leibniz's word are there summarized.

[27] Pascal's chief work on centers of gravity is in connection with the cycloid, and solids of revolution formed from it. His method was founded on the indivisibles of Cavalieri. His work was issued as a challenge to contemporaries under the assumed name of Amos Dettonville, and under the same name he published his own solutions, after solutions had been given by Huygens, Wallis, Wren and others.

[28] The method of *ductus plani in planum*, the leading or multiplication of a plane into a plane, employed by Gregory St. Vincent in the seventh book of his *Opus Geometricum* (1649) is practically on the same fundamental principle as the present method of finding the volume of a solid by integration. A simple explanation may be given by means of the figure of a quarter of a cone. Let AOBC be the quarter of a circular cone as Fig. A of which OA is the axis, and ABC the base, so that all sections, such as *abc*, are parallel to ABC and perpendicular to the plane AOC. Let *ad* be the height of a rectangle equal in area to the quadrant *abc*, so that *ad* is the average height of the

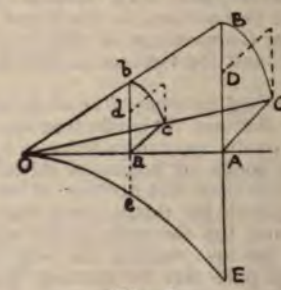


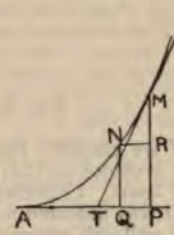
Fig. A.

variable plane *abc*; then the volume of the figure is found by multiplying the height of the variable plane as it moves from O to the position ABC by the corresponding breadth of the plane OAC, i. e., by *bc*, and adding the results.

As we shall see later, Leibniz does not fully appreciate the real meaning of the method; on the other hand Wallis uses the method with good effect in his *Arithmetica Infinitorum*, and states that he has come to it independently. In the above case he would have stated that the product in each case was proportional to the square on *ac*, drawn an ordinate *ae* at right angles to O*a*, so that *ae* represented the product, and so formed the parabola O*e*EA*a*O, of which the area is known to him. This area is proportional to the volume of the cone.

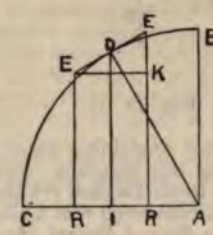
[29] *Ungulae* denote hoof-shaped solids, such as the frusta of cylinders or cones cut off by planes that are not parallel to one another.

[30] The figure here given is of extreme interest. First of all it is not Barrow's "differential triangle," which is that of Fig. B below; this of course is only what those who believe Leibniz's statement that he received no help from Barrow, would expect. By the way, the figure given by Cantor as Barrow's is not quite accurate. (Cantor, III, p. 135.)



BARROW

Fig. B.



PASCAL

Fig. C.

But neither is it the figure of Pascal, which is that of Fig. C. Of course, I am assuming that Gerhardt has given a correct copy of the figure given by Leibniz in his manuscript; although that which I have given of it, a faithful copy of Gerhardt's, shows that his curve was not a circle. I also assume that Cantor is correct in the figure that he gives from Pascal; although Cantor says that the figure occurs in a tract on the sines of a quadrant, and not, as Leibniz states, in a problem on *the measurement of the sphere*. Indeed it seems to me that the figure is more likely to be connected with the area of the zone of a sphere and the proof that this is equal to the corresponding belt on the circumscribing cylinder than anything else. I am bound to assume these things, for I have not had the opportunity of seeing either of the figures in the original for myself. It is strange, in this connection, that Gerhardt in one place (G. 1848, p. 15) gives 1674 as the date of the publication of Barrow, and in another place (G. 1855, p. 45) seven years later, he makes it 1672, and neither of them are correct as the date of the copy that Leibniz could possibly have purchased, namely 1670. This is culpable negligence in the case of a date upon which an argument has to be founded, for one can hardly suspect Gerhardt of deliberate intent to confuse. Nevertheless, like De Morgan, I should have felt more happy if I could have given facsimiles of Barrow's book, and Leibniz's manuscript and figure.

Lastly, there is in Barrow (what neither Gerhardt, Cantor, nor any one else, with the possible exception of Weissenborn, seem to have noticed) chapter and verse for Leibniz's "characteristic triangle." Fig. D is the diagram that Barrow gives to illustrate the first theorem of Lecture XI. This is of course, as is usual with Barrow, a complicated diagram drawn to do duty for a whole set of allied theorems.

In the proof of the first of these theorems occur these words:

"Then the triangle HLG is similar to the triangle PDH (for, on account of the infinite section, the small arc HG can be considered as a straight line).

Hence, HL : LG = PD : DH, or HL . DH = LG . PD,
i. e., HL . HO = DC . Dψ.

By similar reasoning, it may be shown that, since the triangle GMF is similar to the triangle PCG,...."

If now the lines in italics are compared with that part of the figure to which they refer, which has been abstracted in Fig. E, the likeness to Leibniz's figure wants some explaining away, if we consider that Leibniz had the opportunity for seeing this diagram. Such evidence as that would be enough to hang a man, even in an English criminal court. (Further, see note 46.)

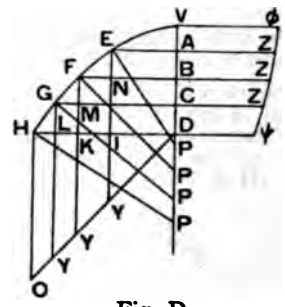


Fig. D.

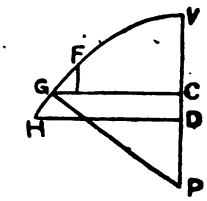


Fig. E.

To sum up, I am convinced that Leibniz was indebted to both of Barrow's diagrams, and also to that of Pascal (for I will call attention to the fact that he uses all three, as I come to them) and I think that after the lapse of thirty years he really could not tell from whom he got his figure. In such a case it would be only natural, if he knew that it was from one of two sources and he was accused of plagiarizing from the one, that he should assert that it was from the other. Hence, by repetition, he would come to believe it. But even this does not explain his letter to d'Hôpital, where he says that he has not obtained any assistance from his methods; unless again we remember that this letter is dated 1694, twenty years after the event.

[31] Great importance, in my opinion hardly merited, is attached to the use by Leibniz of the phrase *momento ex axe* in this place, and in his manuscripts under the heading *Analysis Tetragonistica ex Centrobarycis*, dated October, 1675.

The Latin word *momentum*, a contraction of *movimentum*, has a primary meaning of movement or alteration, and a secondary meaning of a cause producing such movement. The present use of the term to denote the tendency of a force to produce rotation is an example of the use of the word to denote an effect; from the second idea, we have first of all its interpretation as something just sufficient to cause the alteration in the swing of a balance (where the primary idea still obtains), hence something very small, and especially a very small element of time.

Thus we see that Leibniz uses the term in its primary sense, for he employs it in connection with a method *ex Centrobarycis*, and in its mechanical sense, and it is thus fairly justifiable to assume that he got the term from Huygens; in just this sense we now speak of the moment of inertia.

Newton's use of the term is given in Lemma II of Book II of the *Principia*, in the following way.

"I shall here consider such quantities as undetermined or variable, as it were increasing or decreasing by a continual motion or flow (*fluxus*); and their instantaneous (*momentanea*) increments or decrements I shall denote (*intelligo* = understand) by the name "moments"; so that increments stand for moments that are added or positive (*affirmativis*), and decrements for those that are subtracted or negative."

This has nothing whatever to do with what Leibniz means by a moment,

and it seem ridiculous to bring forward the use of this word as evidence that Leibniz had seen Newton's work, or even heard of it through Tschirnhaus, before the year 1675.

The fact that in another place, where I will refer to it again, he uses the phrase "instantaneous increment" is quite another matter.

The use of the word moment in this mechanical sense is here perfectly natural. See Cantor, III, p. 165; also Cantor, II, p. 569, where the *idea* is referred back at least to Benedetti (1530-1590); but the idea is fundamental in the theorems due to Pappus concerning the connection between the path of the center of gravity of an area and the surfaces and volumes of rings generated by the area, of which the proofs were given by Cavalieri. When, however, and by whom, the *word* moment was itself first used in this connection, I have been unable to find the slightest trace.

[32] With due regard to the statement that Leibniz "had looked through Cavalieri" before he went to Paris, it is not remarkable that he did not notice very much at all in Cavalieri. Cavalieri's *Geometria indivisibilibus* is not a book to be "looked through." It is a work for weeks of study. I cannot say whether the idea involved in Leibniz's characteristic triangle is used by Cavalieri as such; but I do not see how else he could have given proofs (as stated by Williamson in his article on "Infinitesimal Calculus" in the *Times* edition of the *Encyc. Brit.*) of Pappus's theorem for the area of a ring; and I should think that it is morally certain that Cavalieri is the source from which Wallis obtained his ideas for the rectification of the arc of the spiral. I had occasion to refer to a copy in the Cambridge University Library, and what I saw of it in the short time at my disposal determined me to make a translation of it, with a commentary, *as soon as I had enough time at my disposal.* "As one reads tales of romance"!

[33] Note that this is proportional to the area of the surface formed by the revolution of the curve C(C) about the axis AP. Barrow does not use the method to find the areas of surfaces of revolution; he prefers to straighten out the curve C(C), and erect the ordinates BC, (B)(C) perpendicular to the curve thus straightened; i. e., he works with the product BC.C(C) as it stands. But, after giving the determination of the surface of a right circular cone as an example of the method, and as a means of combating the objections of Tacquet to the method of indivisibles, he goes on to say: "Evidently in the same manner we can investigate most easily the surfaces of spheres and portions of spheres (nay, provided all necessary things are given or known, any other surfaces that are produced in this way). But I propose to keep, to a great extent, to more general methods" (end of Lecture II). Thus we find that Barrow does not give any further examples of the determination of the areas of surfaces of revolution until Lecture XII. And why? Because he is not writing a work on mensuration, but a calculus. The reference to the method of indivisibles however shows that in Barrow's opinion, if Cavalieri had not used his method for the determination of the area of the surface of a sphere, then he ought to have done so.

[34] It is difficult to see also how Huygens could have performed his constructions unless he had used the method that Leibniz claims to have discovered.

[35] It is strange that Roberval, as an independent discoverer of the method of indivisibles, did not perceive the method of the constructions of Huygens. Of Bullialdus I can find no mention except as the author of a set of navigation tables. Cantor does not even refer to him, as far as I can find.

[36] This conversation probably took place late in 1673; see a note on the alteration of the date of a manuscript dated November 11, 1673, where the 3 was originally a 5 (Section below).

The method of Slusius (de Sluze, or Sluse) is as follows:

Suppose that the equation of the given curve is

$$x^3 - 2x^2y + bx^2 - b^2x + by^2 - y^3 = 0.$$

Slusius takes all the terms containing y, multiplies each by the corresponding index of y; then similarly takes all the terms containing x, multiplies each by the corresponding index of x, and divides each term of the result by x; the quotient of the former by the last expression gives the value of the subtangent. This is practically the content of Newton's method of *analysis per aequationes*, and Slusius sent an account of it to the Royal Society in January, 1673. It was printed in the *Phil. Trans.*, as No. 90. This is given by Gerhardt (G. 1848, p. 15) as an example of the method of Slusius. It is rather peculiar that Gerhardt does not mention that this is the example given by Newton in the oft-quoted letter of December 10, 1672, and represents what Newton "guesses the method to be." As it stands in G. 1848, it would appear to be a quotation from the work of Slusius himself. There is evidence that Leibniz had seen the explanation given in the *Phil. Trans.*, or had been in communication with Slusius; this will be referred to later, but it may be said here that this fact makes Leibniz somewhat independent of any necessity of having seen Newton's letter.

[37] Some point is made of the question why, if Leibniz had seen the "differential triangle" of Barrow, he should have called it by a different name. If there were any point in it at all, it would go to prove that Barrow's calculus was published by Barrow as a *differential* calculus. But there is no point, for Barrow never uses the term! It is a product of later growth, by whom first applied I know not. Leibniz, thus free to follow his logical plan of denominating everything, uses a term borrowed from his other work. He thus defines a character or characteristic. "Characteristics are certain things by means of which the mutual relations of other things can be expressed, the latter being dealt with more easily than are the former." See Cantor, III, p. 33f.

[38] Gregory's *Geometriae Pars Universalis* was published at Padua in 1668. Leibniz had either this book, or the Barrow in which one of Gregory's theorems is quoted, close at hand in his work. For he gives it as an example of the power of his calculus, referring to a diagram which is not drawn. This diagram I was unable to draw from the meager description of it given by Leibniz, until *I looked up Barrow's figure*, in default of being able to obtain a copy of Gregory's work; thereupon the figure was drawn immediately.

[39] Here indeed it must be admitted that Leibniz is —suffering from a lapse of memory. As has been said before, Barrow's lectures appeared in 1670 and were in the possession of Leibniz before ever he dreamed of his theorems. But what can one expect when admittedly this account (from which the *Historia* was in all probability written up) is purely from memory, aided by the few manuscripts that he had kept. Gerhardt does not say that he has found, nor does he publish, any manuscripts that could possibly give the order in which the text-books that Leibniz procured were read. Which of us, at the age of 57, could say in what order we had read books at the age of 27; or, if by then we had worked out a theory, could with accuracy describe the steps by which we climbed, or from a mass of muddle and inaccuracies, say to whom we were indebted for the first elementary ideas that we had improved beyond all recognition? I doubt whether any of us would recognize our own work under such circumstances.

[40] Again Leibniz makes a bad mistake in affecting to despise the work of his rivals—for that is what the words, "these things were perfectly easy to the veriest beginner who had been trained to use them," makes us believe. It is also bad taste, for, besides Barrow, Huygens also remained true to the method of geometry till his death. The sentence which follows is "pure swank," and as a matter of fact it was left to others, such as the Bernoullis, to make the best use of the method of Leibniz. The great thing we have to thank Leibniz for is the notation; it is a mistake to call this the invention of a notation for the infinitesimal calculus. As we shall see, Leibniz invented this notation for finite differences, and only applied it to the case in which the differences were infinitely small. Barrow's method, of a and e, also survives to the present day, under the disguise of h and k, in the method by which the

elements of the calculus are taught in nine cases out of ten. For higher differential coefficients the suffix notation is preferable, and later on the operator D is the method *par excellence.*

[41] Here Leibniz seems to be unable to keep from harking back to the charge made by Fatio, suggesting that by the publication of his letters by Wallis this charge has been proved to be absolutely groundless.

[42] It is probable that this may mean "has received high commendation"; for *elogiis* may be the equivalent of eulogy, in which case *celebratus est* must be translated as "has been renowned."

[43] This is untrue. As has been said, the attack was first made publicly in 1699; at this time, although Huygens had indeed been dead for four years, Tschirnhaus was still alive, and Wallis was appealed to by Leibniz. It is strange that Leibniz did not also appeal to Tschirnhaus, through whom it is suggested by Weissenborn that Leibniz may have had information of Newton's discoveries. Perhaps this is the reason why he did not do so, since Tschirnhaus might not have turned out to be a suitable witness for the defense. Leibniz must have had this attack by Fatio in his mind, for he could hardly have referred to Keill as a *novus homo,* while we know that he did not think much of Fatio as a mathematician. To say that there never existed any uncertainty as to the name of the true inventor until 1712 is therefore sheer nonsense; for if by that he means to dismiss with contempt the attack of Fatio, who can he mean by the phrase *novus homo*? The sneering allusion to "the hope of gaining notoriety by the discussion" can hardly allude to any one but Fatio. Finally if Fatio is dismissed as contemptible, the second attack by Keill was made in 1708. If it was early in the year, Tschirnhaus was even then alive, though Wallis was dead.

[44] Gerhardt says in a note (G. 1846, p. 22) that his real name was probably Kramer; for what reason I am unable to gather. Cantor says distinctly that his name was Kaufmann, and this is the usually accepted name of the man who was one of the first members of the Royal Society and contributed to its *Transactions.* It seems to me that Gerhardt is guessing; the German word *Kramer* means a small shopkeeper, while *Kaufmann* means a merchant. To Mercator is due the logarithmic series obtained by dividing unity by $(1+x)$ and integrating the resulting series term by term; the connection with the logarithm of $(1+x)$ is through the area of the rectangular hyperbola $y(1+x)=0$. See Reiff, *Geschichte der unendlichen Reihen.*

[45] Newton obtained the general form of the binomial expansion after the method of Wallis, i. e., by interpolation. See Reiff.

[46] We now see what was Leibniz's point; the differential calculus was not the employment of an infinitesimal and a summation of such quantities; it was the use of the idea of these infinitesimals being differences, and the employment of the notation invented by himself, the rules that governed the notation, and the fact that differentiation was the inverse of a summation; and perhaps the greatest point of all was that the work had not to be referred to a diagram. This is on an inestimably higher plane than the mere differentiation of an algebraic expression whose terms are simple powers and roots of the independent variable.

[47] Why is Barrow omitted from this list? As I have suggested in the case of Barrow's omission of all mention of Fermat, was Leibniz afraid to awake afresh the sleeping suggestion as to his indebtedness to Barrow? I have suggested that Leibniz read his Barrow on his journey back from London, and perhaps, tiring at having read the Optics first and then the preliminary five lectures, just glanced at the remainder and missed the main important theorems. I also make another suggestion, namely, that perhaps, or probably, in his then ignorance of geometry he did not understand Barrow. If this is the case it would have been gall and wormwood for Leibniz to have ever owned to it. Then let us suppose that in 1674 with a fairly competent knowl-

edge of higher geometry he reads Barrow again, skipping the Optics of which he had already formed a good opinion, and the wearisome preliminary lectures of which he had already seen more than enough. He notes the theorems as those he has himself already obtained, and the few that are strange to him he translates into his own symbolism. I suggest that this is a feasible supposition, which would account for the marks that Gerhardt states are made in the margin. It would account for the words "in which latter I found the greater part of my theorems anticipated" (this occasion in future times ranking as the first time that he had really read Barrow, and lapse of memory at the end of thirty years making him forget the date of purchase, possibly confusing his two journeys to London); it would account for his using Barrow's differential triangle instead of his own "characteristic triangle." As Barrow tells his readers in his preface that "what these lectures bring forth, or to what they may lead you may easily learn from *the beginnings of each*," let us suppose that Leibniz took his advice. What do we find? The first four theorems of Lecture VIII give the geometrical equivalent of the differentiation of a power of *a dependent* variable; the first five of Lecture IX lead to a proof that, expressed in the differential notation,

$$(ds/dx)^2 = 1 + (dy/dx)^2;$$

the appendix to this lecture contains the differential triangle, and five examples on the *a* and *e* method, fully worked out; the first theorem in Lecture XI has a diagram such that, when that part of it is dissected out (and Barrow's diagrams want this in most cases) which applies to a particular paragraph in the proof of the theorem, this portion of the figure is a *mirror image of the figure* drawn by Leibniz when describing the characteristic triangle (turn back to note 30). I shall have occasion to refer to this diagram again. The appendix to this lecture opens with the reference to the work of Huygens; and the second theorem of Lecture XII is the strangest coincidence of all. This theorem in Barrow's words is:

"Hence, if the curve AMB is rotated about the axis AD, the ratio of the surface produced to the space ADLK is that of the circumference of a circle to its diameter; whence, if the space ADLK is known, the said surface is known."

The diagram given by Barrow is as usual very complicated, serving for a group of nine propositions. Fig. F is that part of the figure which refers to the theorem given above, dissected out from Barrow's figure. Now remember that Leibniz always as far as possible kept his axis clear on the left-hand side of his diagram, while Barrow put his datum figure on the left of his axis, and his constructed figures on the right; then you have Leibniz's diagram and the proof is by the similarity of the triangles MNR, PMF, where

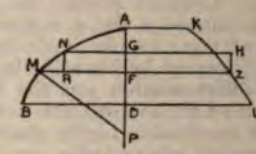


Fig. F.

FZ = PM; and the theorem itself is only another way of enunciating the theorem that Leibniz states he generalized from Pascal's particular case! Lastly, the next theorem starts with the words: "Hence the surfaces of the sphere, both the spheroids and the conoids receive measurement." What a coincidence!

As this note is getting rather long, I have given the full proof of the first two theorems of Barrow's Lecture XII as an appendix, at the end of this section.

The sixth theorem of this lecture is the theorem of Gregory which Leibniz also gives later; I will speak of this when I come to it. As also, when we

discuss Leibniz's proof of the rules for a product, etc., I will point out where they are to be found in Barrow ready to his hand.

Yet if all this were so, he could still say with perfect truth that, in the matter of the invention of the differential calculus (as he conceived the matter to consist, that is, the differential and integral notations and the method of analysis), he derived no assistance from Barrow. In fact, once he had absorbed his fundamental ideas, Barrow would be less of a help than a hindrance.

[48] Apollonian geometry comprised the conic sections or curves of the second degree according to Cartesian geometry; curves of a higher degree and of a transcendent nature, like the spiral of Archimedes, were included under the term "mechanical."

[49] The great discovery of Descartes was not simply the application of geometry; that had been done in simple cases ages before. Descartes recognized the principle that every property of the curve was included in its equation, if only it could be brought out. Thus Leibniz's greatest achievement was the recognition that the differential coefficients were also functions of the abscissa. The word functon was applied to certain straight lines dependent on the curve, such as the abscissa itself, the ordinate, the chord, the tangent, the perpendicular, and a number of others (Cantor, III, preface, p. v). This definition is from a letter to Huygens in 1694. There is therefore a great advance made by 1714, the date of the *Historia*, since here it is at least strongly hinted that Leibniz has the algebraical idea of a function.

[50] With regard to Newton, at least, this is untrue. Without a direct reference to the original manuscript of Newton it is quite impossible to state whether even Newton *wrote* 0 or *o*; even then there may be a difficulty in deciding, for Gerhardt and Weissenborn have an argument over the matter, while Reiff prints it as 0. However this may be there is no doubt that Newton considered it as an infinitely small *unit of time*, only to be put equal to zero when it occurred as a factor of terms in an expression in which there also occurred terms that did not contain an infinitesimally small factor. This was bound to be the case, since Newton's $\dot{x}$ and $\dot{y}$ were velocities. In short, expressing Newton's notation in that of Leibniz, we have

$$\dot{x}o \text{ or } \dot{x}0 = (dx/dt).\ dt$$

and therefore xo is an infinitesimal or a differential equal to Leibniz's dx.

[51] This is in a restricted sense true. No one seems to have felt the need of a second differentiation of an original function; those who did differentiated once, and then worked upon the function thus obtained a second time in the same manner as in the first case. Barrow indeed only considered curves of continuous curvature, and the tangents to these curves; but Newton has the notation $\ddot{x}$, etc. But the idea had been used by Slusius in his *Mesolabum* (1659), where a general method of determining points of inflection is made to depend on finding the maximum and minimum values of the subtangent. Lastly, it can hardly be said that Leibniz's interpretation of $\int\int$ ever attained to the dignity of a double integral in his hands.

[52] David Gregory is not the only sinner! Leibniz, using his calculus, makes a blunder over osculations, and will not stand being told about it; he simply repeats in answer that he is right (Rouse Ball's *Short History*).

[53] The names of the committee were not even published with their report. In fact the complete list was not made public until De Morgan investigated the matter in 1852! For their names see De Morgan's *Newton*, p. 27.

[54] What then made Leibniz change his mind?

[55] It is established that this was Johann (John) Bernoulli; see Cantor, III, p. 313f; Gerhardt gives a reference to Bossut's *Geschichte*, Part II, p. 219.

[56] This seems to be an intentional misquotation from Bernoulli's letter, which stated that Newton did not understand the meaning of higher differen-

tiations. At least, that is what Cantor says was given in the pamphlet; and it is established that....

[57] the pamphlet referred to was also an anonymous contribution by Leibniz himself! Is it strange that hard things are both thought and said of such a man?

[58] Again this is Leibniz himself! Had he then no friends at all to speak for him and dare subscribe their signatures to the opinion? Unfortunately Tschirnhaus was dead at the time of the publication of the *Commercium Epistolicum*, but he could have spoken with overwhelming authority, as Leibniz's co-worker in Paris, at any time between the date of Leibniz's review of Newton's *De Quadratura* in the *Acta Eruditorum* until his death in 1708, even if he had died before the publication of Keill's attack in the *Phil. Trans.* of that year was made known to him. Does not this silence on the part of Tschirnhaus, the personal friend of Leibniz, rather tend to make Leibniz's plea, that his opponents had had the shrewdness to wait till Tschirnhaus, among others, was dead, recoil on his own head, in that he has done the very same thing? Leibniz must have known the feeling that this review aroused in England, and, Huygens being dead, Tschirnhaus was his only reliable witness. Of course I am not arguing that Leibniz did found his calculus on that of Newton. I am fully convinced that they both were indebted to Barrow, Newton being so even more than Leibniz, and that they were perfectly independent of one another in the development of the *analytical* calculus. Newton, with his great knowledge of and inclination toward geometrical reasoning, backed with his personal intercourse with Barrow, could appreciate the finality of Barrow's proofs of the differentiation of a product, quotient, power, root, logarithm and exponential, and the trigonometrical functions, in a way that Leibniz could not. But Newton never seems to have been accused of plagiarism from Barrow; even if he had been so accused, he probably had ready as an answer, that Barrow had given him permission to make any use he liked of the instruction that he obtained from him. Leibniz, when so accused, replied by asserting, through confusion of memory I suggest, that he got his first idea from the works of Pascal. Each developed the germ so obtained in his own peculiar way; Newton only so far as he required it for what he considered his main work, using a notation that was of greatest convenience to him, and finally falling back on geometry to provide himself with what appealed to him as rigorous proof; Leibniz, more fortunate in his philosophical training and his lifelong effort after symbolism, has ready to hand a notation, almost developed and perfected when applied to finite quantities, which he saw with the eye of genius could be employed as usefully for infinitesimals. De Morgan justly remarks that one dare not accuse either of these great men of deliberate untruth with regard to specific facts; but it must be admitted that neither of them can be considered as perfectly straightforward; and the political similitude, which Cantor speaks of, in which nothing is too bad to be said of an opponent, seems to have applied just as much to the mathematician of the day as to the politician.

[59] This was given in more detail in the first draught of this essay (G. 1846, p. 26): Hitherto, while still a pupil, he kept trying to reduce logic itself to the same state of certainty as arithmetic. He perceived that occasionally from the first figure there could be derived a second and even a third, without employing conversions (which themselves seemed to him to be in need of demonstration), but by the sole use of the principle of contradiction. Moreover, these very conversions could be proved by the help of the second and third figures, by employing theorems of identity; and then now that the conversion had been proved, it was possible to prove a fourth figure also by its help, and this latter was thus more indirect than the former figures. He marveled very much at the power of identical truths, for they were generally considered to be useless and nugatory. But later he considered that the whole of arithmetic and geometry arose from identical truths, and in general that all undemonstrable truths depending on reasoning were identical, and that these combined

with definitions yield identical truths. He gave as an elegant example of this analysis a proof of the theorem, The whole is greater than its part.

[60] It is fairly certain that Leibniz could not possibly at this time have perceived that in this theorem he has the germ of an integral. The path to the higher calculus lay through geometry. As soon as Leibniz attained to a sufficient knowledge of this subject he would recognize the area under a curve between a fixed ordinate and a variable one as a set of magnitudes of the kind considered, the ordinates themselves being the differences of the set; he would see that there was no restriction on the number of steps by which the area attained its final size. Hence, in this theorem he has a proof to hand that integration as a determination of an area is the inverse of a difference. This does not mean the inverse of a differentiation, i. e., the determination of a *rate*, or the drawing of a tangent. As far as I can see, Leibniz was far behind Newton in this, since Newton's fluxions were founded on the idea of a rate.

[61] It is a pity that we are not told the date at which Leibniz read his Wallis; it is a greater pity that Gerhardt did not look for a Wallis in the Hanover Library and see whether it had the date of purchase on it (for I have handled lately several of the books of this time, and in nearly every case I found inserted on the title page the name of the purchaser and the date of purchase). I make this remark, because there arises a rather interesting point. Wallis, in his *Arithmetica Infinitorum*, takes as the first term of all his series the number 0, and in one case he mentions that the differences of the differences of the cubes is an arithmetical series. He also works out fully the sums of the figurate numbers (or as Leibniz calls them the combinatory numbers), the general formulas for these sums he calls their *characteristics*. He also remarks on the fact that any number can be obtained by the addition of the one before it and the one above it (which is itself the sum of all the numbers in the preceding column above the one to the left of that which he wishes to obtain). Thus, in the fourth column 4 is the sum of 3 (to the left) and I (above), i. e., the sum of the two first numbers in column three; 10 is the sum of 6 (to the left) and 4 (above, which has been shown to be the sum of the first two numbers of column three), and therefore 10 is the sum of the first three numbers in column three. Now my point is, assuming it to have been impossible that Leibniz had read Wallis at the time that he was compiling his *De Arte*, we have here another example, free from all suspicion, of that series of instances of independent contemporary discoveries that seems to have dogged Leibniz's career.

[62] The name surdesolid to denote the fifth power is used by Oughtred, according to Wallis. By Cantor the invention of the term seems to be credited to Dechales, who says, "The fifth number from unity is called by some people the quadrato-cubus, but this is ill-done, since it is neither a square nor a cube and cannot thus be called the square of a cube nor the cube of a square: we shall call it supersolidus or surde solidus." (Cantor, III, p. 16.)

[63] This theorem is one of the fundamental theorems in the theory of the summation of series by finite differences, namely,

$$\Delta^m u_n = u_{n+m} - {}_mC_1 \cdot u_{n+m-1} + {}_mC_2 \cdot u_{n+m-2} - \text{etc.},$$

which is usually called the direct fundamental theorem; for although Leibniz could not have expressed his results in this form since he did not know the sums of the figurate numbers as generalized formulas (or I suppose not, if he had not read Wallis), and apparently his is only a general case, yet it must be remembered that any term of the first series can be chosen as the first term. It is interesting to note that the second fundamental theorem, the inverse fundamental theorem, was given by Newton in the *Principia*, Book III, lemma V, as a preliminary to the discussion on comets at the end of this book. Here he states the result, without proof, as an interpolation formula; (it is frequently referred to as Newton's Interpolation Formula); it may however be used as an extrapolation formula, in which case we have

$$u_{m+n} = u_m + {}_nC_1 \cdot \Delta u_m + {}_nC_2 \cdot \Delta^2 u_m + \text{etc.}$$

In the two formulas as given here, the series are

$$u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad \text{etc.}$$
$$\Delta u_1 \quad \Delta u_2 \quad \Delta u_3 \quad \Delta u_4 \quad \text{etc.}$$
$$\Delta^2 u_1 \quad \Delta^2 u_2 \quad \Delta^2 u_3 \quad \text{etc. and so on.}$$

[64] What are we to understand by the inclusion of this series in this connection? Does Leibniz intend to claim this as his? I have always understood that this is due to John Bernoulli, who gave it in the *Acta Eruditorum* for 1694, in a slightly different form, and proved by direct differentiation; and that Brook Taylor obtained it as a particular case of a general theorem in and by *finite differences*. If Leibniz intended to claim it, he has clearly anticipated Taylor. It is quite possible that Leibniz had done so, even in his early days; and as soon as in 1675, or thereabouts, he had got his signs for differentiation and integration, it is possible that he returned to this result and expressed it in the new notation; for the theorem follows so perfectly naturally from the last expression given for $a - \omega$. But it is hardly probable, for Leibniz would almost certainly have shown it to Huygens and mentioned it.

The other alternative is that here he is showing how easily Bernoulli's series could have been found in a much more *general form*, i. e., as a theorem that is true (as he indeed states) for finite differences as well as for infinitesimals; the inclusion of this statement makes it very probable that this supposition is a correct one. This leads to a pertinent, or impertinent, question. Brook Taylor's *Methodus Incrementorum* was published in 1715; the *Historia* was written some time between 1714 and 1716; Gerhardt states that there were two draughts of the latter, and that he is giving the second of these. In justice to Leibniz there should be made a fresh examination of the two draughts, for if this theorem is not given in the original draught it lays Leibniz open to further charge of plagiarism. I fully believe that the theorem will be found in the first draught as well and that my alternative suggestion is the correct one.

In any case, the tale of the *Historia* is confused by the interpolation of the symbolism invented later (as Leibniz is careful to point out). The question is whether this was not intentional. And this query is not impertinent, considering the manner in which Leibniz refrains from giving dates, or when we compare the essay in the *Acta Eruditorum*, in which he gives to the world the description of his method. Weissenborn considers that "this is not adapted to give an insight into his methods, and it certainly looks as if Leibniz wished deliberately to prevent this." Cf. Newton's "anagram" (sic), and the Geometry of Descartes, for parallels.

[65] In reference to the employment of the calculus to diagrammatic geometry, as will be seen later, Leibniz says:

"But our young friend quickly observed that the differential calculus could be employed with figures in an even more wonderfully simple manner than it was with numbers, because *with figures the differences were not comparable with the things which differed*; and as often as they were connected together by addition or subtraction, being incomparable with one another, the less vanished in comparison with the greater."

[66] This makes what has just gone before date from the time previous to his coming across Cavalieri. See note following.

[67] This is about the first place in which it is possible to deduce an exact date, or one more or less exact. According to Leibniz's words that immediately follow it may be deduced that it was somewhere about twelve months before the publication of the Hypothesis of Physics—if we allow for a slight interval between the dropping of the geometry and the consideration of the principles of physics and mechanics, and a somewhat longer interval in which to get together the ideas and materials for his essay—that he had finished his

"slight consideration" of Leotaud and Cavalieri. This would make the date 1670, and his age 24.

[68] This essay founded the explanation of all natural phenomena on motion, which in turn was to be explained by the presence of an all-pervading ether; this ether constituted light.

[69] The dedication of the *Nova methodus* in 1667 to the Elector of Mainz (ancient name Moguntiacum) procured for Leibniz his appointment in the service of the latter, first as an assistant in the revision of the statute-book, and later on the more personal service of maintaining the policy of the Elector, that of defending the integrity of the German Empire against the intrigues of France, Turkey and Russia, by his pen.

[70] This probably refers to the time when his work on the statute-book was concluded, and Leibniz was preparing to look for employment elsewhere.

[71] This is worthy of remark, seeing that Leibniz had attempted to explain gravity in the *Hypothesis physica nova* by means of his concept of an ether. The conversation with Huygens had results that will be seen later in a manuscript (§ III below) where Leibniz obtains quadratures *"ex Centrobarycis."* It also probably had a great deal to do with Leibniz's concept of a "moment."

[72] The use of the word *veterno*—which I have translated "lethargy" as being the nearest equivalent to the fundamental meaning, the sluggishness of old age—coupled with his remark that he was in no mind to enter fully into these more profound parts of mathematics, sheds a light upon the reason why he had so far done no geometry. Also the last words of the sentence give the stimulus that made him cast off this lethargy; namely, shame that he should appear to be ignorant of the matter. This would seem to be one of the great characteristics of Leibniz, and might account for much, when we come to consider the charges that are made against him.

[73] We have here a parallel (or a precedent) for my suggestion that Leibniz was mentally confusing Barrow and Pascal as the source of his inspiration for the characteristic triangle. For here, without any doubt whatever, is a like confusion. What Pell told him was that his theorems on numbers occurred in a book by Mouton entitled *De diametris apparentibus Solis et Lunae* (published in 1670). Leibniz, to defend himself from a charge of plagiarism, made haste to borrow a copy from Oldenburg and found to his relief that not only had Mouton got his results by a different method, but that his own were more general. The words in italics are interesting.

Of course these words are not italicized by Gerhardt, from whom this account has been taken (G. 1848, p. 19); nor does he remark on Leibniz's lapse of memory in this instance. Further there is no mention made of it in connection with the *Historia*, i. e., in G. 1846. Is it that Gerhardt, as counsel for the defense, is afraid of spoiling the credibility of his witness by proving that part of his evidence is unreliable? Or did he not become aware of the error till afterward? See Cantor, III, p. 76.

[74] An instance is referred to on p. 85 of De Morgan's *Newton*, showing the sort of thing that was done by the committee. This however is not connected with a letter to Oldenburg, but to Collins. It may be taken as a straw that shows the way the wind blew.

[75] Observe that nothing has been said of the fact that Leibniz had purchased a Barrow and took it back with him to Paris.

[76] Cf. the remark in the postscript to Bernoulli's letter, where Leibniz says that the work of Descartes, looked at at about the same time as Clavius, that is, while he was still a youth, "seemed to be more intricate."

[77] The *libellus* referred to would seem to be the work on the cycloid, written by Pascal in the form of letters from one Amos Dettonville to one M. de Carcavi.

[78] This theorem is given, and proved by the method of indivisibles, as

Theorem I, of Lecture XII in Barrow's *Lectiones Geometricae*; and Theorem II is simply a corollary, in which it is remarked:

"Hence the surfaces of the sphere, both the spheroids, and the conoids receive measurement. . . ."

The proof of these two theorems is given at the end of the section as a supplement. See also Note 46, for its significance.

[79] The whole context here affords suggestive corroboration in favor of the remarks made in Note 31 on the use of the word "moment," though the connection with the determination of the center of gravity is here overshadowed by its connection with the surface formed by the rotation of an arc about an axis.

[80] The figure given is exactly that given by Gerhardt, with the unimportant exception that, for convenience in printing, I have used U instead of Gerhardt's Θ, a V instead of his ת (a Hebrew T), and a Q for his Π. I take it, of course, that Gerhardt's diagram is an exact transcript of Leibniz's, and it is interesting to remark that Leibniz seems to be endeavoring to use T's for all points on the tangent, and P's for points on the normal, or perpendicular, as it is rendered in the Latin.

This diagram should be compared with that in the "postscript" written nine or ten years before. Note the complicated diagram that is given here. and the introduction of the secant that is ultimately the tangent, which does not appear in the first figure. From what follows, this is evidently done in order to introduce the further remarks on the similar triangles. It adds to the confusion when an effort is made to determine the dates at which the several parts were made out. For instance, the remark that finite triangles can be found similar to the characteristic triangle probably belongs approximately to the date of his reply to the assertions of Nieuwentiit, which will be referred to later.

[81] The notation introduced in the lettering should be remarked. His early manuscripts follow the usual method of the time in denoting different positions of a variable line by the same letter, as in Wallis and Barrow, though even then he is more consistent than either of the latter. He soon perceives the inconvenience of this method, though as a means of generalizing theorems it has certain advantages. We therefore find the notation C, (C), ((C)), for three consecutive points on a curve, as occurs in a manuscript dated (or it should be) 1675. This notation he is still using in 1703; but in 1714, he employs a subscript prefix. This is all part and parcel with his usual desire to standardize and simplify notations.

[82] This sentence conclusively proves that Leibniz's use of the moment was for the purposes of quadrature of surfaces of rotation.

[83] "From these results"—which I have suggested he got from Barrow—"our young friend wrote down a large collection of theorems." These theorems Leibniz probably refers to when he says that he found them all to have been anticipated by Barrow, "when his Lectures appeared." I suggest that the "results" were all that he got from Barrow on his first reading, and that the "collection of theorems" were found to have been given in Barrow when Leibniz referred to the book again, after his geometrical knowledge was improved so far that he could appreciate it.

[84] The use of the first person is due to me. The original is impersonal, but is evidently intended by Leibniz to be taken as a remark of the writer, "the friend who knew all about it." The distinction is marked better by the use of the first personal pronoun than in any other way.

[85] Query, all except Leibniz, the Bernoullis, and one or two others.

[86] Tetragonism = quadrature; the arithmetical tetragonism is therefore Leibniz's value for π as an infinite series, namely,

"The area of a circle, of which the square on the diameter is equal to unity, is given by the series

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}$$

[87] This is clearly original as far as Leibniz is concerned; but the consideration of a polar diagram is to be found in many places in Barrow. Barrow however forms the polar differential triangle, as at the present time, and does not use the rectangular coordinate differential triangle with a polar figure; nor does Wallis. We see therefore that Leibniz, as soon as ever he follows his own original line of thinking, immediately produces something good.

[88] This is evidently a misprint; it is however curious that it is repeated in the second line of the next paragraph. Probably, therefore, it is a misreading due to Gerhardt, who mistakes AZ for the letters XZ, as they ought to be; and has either not verified them from the diagram, or has refrained from making any alteration.

[89] The symbol $\smile$ is here to be read as "and then along the arc to."

[90] Probably refers to Leibniz's work on curvature, osculating circles, and evolutes, as given in the *Acta Eruditorum* for 1686, 1692, 1694. It is to be noted that with Leibniz and his followers the term evolute has its present meaning, and as such was first considered by Huygens in connection with the cycloid and the pendulum. It signified something totally different in the work of Barrow, Wallis and Gregory. With them, if the feet of the ordinates of a curve are, as it were, all bunched together in a point, so as to become the *radii vectores* of another curve, without rupturing the curve more than to alter its curvature (the area being thus halved), then the first curve was called the evolute of the second and the second the involute of the first. See Barrow's *Lectiones Geometricae*, Lecture XII, App. III, Prob. 9, and Wallis's *Arithmetica Infinitorum*, where it is shown that the evolute, in this sense, of a parabola is a spiral of Archimedes.

[91] The colon is used as a sign of division, and the comma has the significance of a bracket for all that follows. It is curious to notice that Leibniz still adheres to the use of xx for x^2, while he uses the index notation for all the higher powers, just as Barrow did; also, that the bracket is used under the sign for a square root, and that too in addition to the vinculum. For an easy geometrical prof of the relation $x = 2z^2/(1+z^2)$, see Note 94.

[92] See Cantor, III, pp. 78-81. Also note the introduction of what is now a standard substitution in integration for the purpose of rationalization.

[93] This term represents what is now generally known as the method of inversion of series. Thus, if we are given

$$x = y + ay^2 + by^3 + cy^4 + \text{etc.},$$

where x and y are small, then $y = x$ is a first approximation; hence since $y = x - ay^2 - by^3 - cy^4 - \text{etc.}$, we have as a second approximation

$$y = x - ax^2;$$

substituting this in the term containing y^2, and the first approximation, $y = x$, in the term containing y^3, we have

$$y = x - a(x - ax^2)^2 - bx^3 = x - ax^2 + (2a^2 - b)x^3,$$

as a third approximation; and so on.

[94] The relation $x = 2z^2/(1+z^2)$ can be easily proved geometrically for the circle; hence, by using the orthogonal projection theorem, Leibniz's result for the central conic can be immediately derived.

Thus suppose that, in the diagrams below, AC is taken to be unity, AU $= z$ and AX $= x$.

Then, in either figure, since the $\triangle$s BYX, CUA are similar,

$$\text{AX} : \text{XB} = \text{AX}.\text{XB} : \text{XB}^2 = \text{XY}^2 : \text{XB}^2 = \text{AU}^2 : \text{CA}^2;$$

hence, for the circle, we have

$$AX : AB = AU^2 : AC^2 + AU^2, \text{ or } x = 2z^2/(1+z^2);$$

and similarly for the rectangular hyperbola

$$AX : AB = AU^2 : AC^2 - AU^2, \text{ or } x = 2z^2/(1-z^2).$$

Applying all the x's to the tangent at A, we have (by division and integration of the right-hand side, term by term, in the same way as Mercator)

$$\text{area AUMA} = 2(z^3/3 \mp z^5/5 + z^7/7 \mp \text{etc.})$$

Now, since the triangles UAC, YXB are similar, $UA.XB = AC.XY$; hence $2\triangle AYC = 2UA.AC \mp UA.AX = 2UA.AC \mp AUMA \mp 2$ seg. AYA,

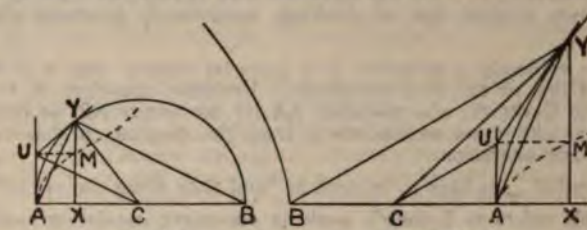


Fig. G. Fig. H.

for Leibniz has shown that AXMA = 2 seg. AYA; hence it follows immediately that

$$\text{sector ACYA} = z \mp z^3/3 + z^5/5 \mp \text{etc.}$$

If now, keeping the vertical axis equal to unity, the transverse axis is made equal to a, Leibniz's general theorem follows at once from the orthogonal projection relation.

Note that z is, from the nature of the diagrams, less than 1.

[95] Wallis's expression for π as an infinite product, given in the *Arithmetica* (or Brouncker's derived expression in the form of an infinite continued fraction), or the argument used by Wallis in his work, could not possibly be taken as a proof that π could not be expressed in recognized numbers.

[96] The letter that is missing would no doubt have been given, in the event of the *Historia* being published. According to Gerhardt it is to be found in *Ch. Hugenii....exercitationes,* ed. Uylenbroeck, Vol. I, p. 6, under date Nov. 7, 1674.

[97] Collins wrote to Gregory in Dec. 1670, telling him of Newton's series for a sine, etc.; Gregory replied to Collins in Feb. 1671, giving him three series for the arc, tangent and secant; these were probably the outcome of his work on *Vera Circuli* (1667).

[98] By Mercator; query, also an allusion to Brouncker's article in the *Phil. Trans.,* 1668.

[99] Quite conclusive; no other argument seems required.

[100] This date, April 12, 1675, is important; it marks the time when Leibniz first began to speak of geometry in his correspondence with Oldenburg, as he says below.

[101] Newton obtained the series for arcs in x from the relation $\dot{a} : \dot{x} = 1 : \sqrt{(1-x^2)}$, by expansion and integration, and then the series for the sine by the "extraction of roots." See Note 93, and, for Newton's own modification, Cantor, III, p. 73.

[102] It would appear from this that Leibniz could differentiate the trigonometrical functions. Professor Love, on the authority of Cantor, ascribes them to Cotes; but I have shown in an article in *The Monist* for April, 1916, that Barrow had explicitly differentiated the tangent and that his figures could be used for all the other ratios.

[103] Probably only to test Leibniz's knowledge.

[104] Gerhardt states that in the first draft of the *Historia*, Leibniz had bordered the Harmonic Triangle, as given here, with a set of fractions, each equal to 1/1, so as to more exactly correspond with the Arithmetical Triangle.

[105] The sign here used appears to be an invention of Leibniz to denote an identity, such as is denoted by ≡ at present.

[106] This, and other formulas of the same kind, had been given by Wallis in connection with the formulas for the sums of the figurate numbers. Wallis called these latter sums the "characters" of the series.

[107] This sentence, in that it breaks the sense from the preceding sentence to the one that follows, would appear to be an interpolated note.

[108] There is an unimportant error here. The first value of x evidently should be 0, and not 1.

[109] Why not? Newton's dotted letters still form the best notation for a certain type of problem, those which involve equations of motion in which the independent variable is the time, such as central orbits. Probably Leibniz would class the suffix notation as a variation of his own, but the D-operator eclipses them all. For beginners, whether scholastic or historically such (like the mathematicians that Barrow, Leibniz and Newton were endeavoring to teach), the separate letter notation has most to recommend it on the score of ease of comprehension; we find it even now used in partial differential equations.

[110] Leibniz does not give us an opportunity of seeing how he would have written the equivalent of $dxdxdx$; whether as dx^3 or $\overline{dx}^3$ or $(dx)^3$.

[111] *Ductus* and *ungulae* have already been explained in Notes 28, 29; *cuneus* denotes a wedge-shaped solid, cf. "cuneiform."

[112] This only proves the proportionality, enabling Leibniz to convert the equation $2\int dy/y = 3\int dx/x$ into $2 \log y = 3 \log x$. It will hardly suffice as it stands to enable him to deal with such an equation as $2\int dy/y = 3\int x\,dx$; and it is to be noted that Leibniz does not notice at all the constant of integration. Although Barrow has differentiated (and therefore also has the inverse integral theorems corresponding thereto) both a logarithm and an exponential in Lecture XII, App. III, Prob. 3, 4, yet these problems are in such an ambiguous form that it may be doubted whether Barrow was himself quite clear on what he had obtained. Hence this clear statement of Leibniz must be considered as a great advance on Barrow.

[113] Almost seems to read as a counter-charge against Newton of stealing Leibniz's calculus. Note the tardy acknowledgement that Barrow has previously done of all that Newton has given.

[114] The whole effect that this *Historia* produces in my mind is that the entire thing is calculated to the same end as the *Commercium Epistolicum*. The pity of it is that Leibniz could have told such a straightforward tale, if events had been related in strict *chronological* order, without any interpolations of results that were derived, or notation that was perfected, later. A tale so told would have proved once and for all how baseless were the accusations of the *Commercium*, and largely explained his denial of any obligations to Barrow.

J. M. CHILD.

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CRITICISMS AND DISCUSSIONS.

CURRENT PERIODICALS.

The number of the *Revue de métaphysique et de morale* for January, 1916, is wholly devoted to the commemoration of Malebranche, whose death took place on October 13, 1715. Maurice Blondel writes on the anti-Cartesianism of Malebranche, Emile Boutroux on the intellectualism of Malebranche, Pierre Duhem on the optical work of Malebranche, R. Thamin on Malebranche's *Traité de morale*, E. van Biéma on how Malebranche conceived psychology, and Victor Delbos on Malebranche and Maine de Biran; while Desiré Roustan puts in a plea for an edition of the collected works of Malebranche.

* * *

Among the especially noteworthy articles in the *Bulletin of the American Mathematical Society* for 1916 are reviews which are wonderful examples of research, by Prof. R. C. Archibald of books on the life and work of Napier, and of mathematical quotations (January number), and of Goldenring's history of the construction of a regular polygon of seventeen sides (February number); Dr. R. L. Moore's article on a non-metrical pseudo-Archimedean axiom (February number); and Prof. E. J. Wilczynski's address on the historical development and the future prospects of the differential geometry of plane curves, in which a precise and profound delimitation of the subject-matter of differential geometry is given.

* * *

There are three papers of particular interest to the readers of *The Monist* in the number of the *Transactions of the American Mathematical Society* for January, 1916: Prof. W. F. Osgood sets at rest some interesting questions in the theory of analytic functions of several complex variables by means of simple examples;

Profs. E. B. Van Vleck and F. H. Doubler study Theta functions as defined by functional equations; and Dr. B. A. Bernstain, starting from *class* and *operation* as primitive ideas, succeeds in reducing to four the number of postulates necessary for Boole's algebra of logic.

* * *

In the number of *Scientia* for February 1916, the Abbé Th. Moreux discusses the problem of the novae—stars which appear suddenly at certain periods in the heavens—and the constitution of the universe. Fillippo Bottazzi gives the second part of his article on the fundamental physiological activities; this part is on muscular activity. Annie S. D. Maunder (Mrs. Walter Maunder) deduces some interesting things about prehistoric Iranian migrations from passages in sacred books of Persia—the *Vendidad* and the *Tir Yasht*. Charles Gide writes on the expenditures of the belligerent nations and their economic consequences; and Achille Loria writes on the probable social and economic consequences of the war. Besides this there are reviews of books and periodicals, and French translations of articles in Italian and English.

In *Scientia* for March, 1916, C. G. Abbot writes on the sun as regards its composition and state as transmitter and receiver of energy. E. Bouty gives the first part of an article on the kinetic theory of gases. This part is devoted to the foundations, and it is interesting to notice that the author says that in a kinetic and therefore mechanistic theory we must consider, besides visible motions, *hypothetical and invisible motions*. Louis Matruchot writes on the light thrown on the problem of cancer by vegetable pathology, cancers having been discovered in vegetables. Otto Jespersen of Copenhagen gives some reflections of a Dane on the war; and Camillo Supino of Pavia writes on the economic sources of the war. The number is completed by book reviews and a review of periodicals.

In *Scientia* for April, 1916, Aldo Mieli writes on the pneumatic period of chemistry: the study of gases from the time of Robert Boyle to that of Lavoisier. E. Bouty gives an account of the development and difficulties of the kinetic theory of gases; the question of thermal radiation will be treated in another article as this subject is a great difficulty in the way of the kinetic theory. Etienne Rabaut writes on embryonic phenomena and phylogenesis. J. Holland Rose of Cambridge, England, discusses the future of

Europe; and C. A. Reuterskiöld of Upsala in Sweden indicates what he thinks should be the chief lines of international law after the war. There is a general review of the problems of the fable with special reference to Hindu literature, by A. M. Pizzagalli. There are also reviews of books and periodicals.

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The number of *Scientia* for June 1916 opens with a suggestive article by Professor Pincherle on "Intuition and the Calculus of Probabilities." The word "probability" or "chance" is meaningless to the man from whom no causes are hidden. The definition of "probability" implies the principle of equivalence of causes, and this symmetrical principle implies the absence of any cause which is even in the smallest degree a dominating cause. The simpler cases with which the calculus of probabilty deals are those in which the number of possible causes is finite. Things become more complicated when that number is no longer finite, or when the possible causes form a continuum, for the elementary definition of probability must now be generalized. The author proceeds to show that there is more than a simple agreement between the data of intuition and the theoretical results of the calculus of probabilities. By the elaboration of a few principles of extreme simplicity, the calculus substitutes, as it were, for these data, propositions frankly deductive in their character. Thus intuition comes into play—first in the preliminary exploration of a question, secondly, in helping us to foresee results, and finally, in detecting from this or that result the weak spot at which the assault of scientific criticism may be most effective. Prof. W. M. Bayliss deals with "Surface Phenomena in Living Structures." He is inclined to think that it may be safely said that the peculiarities of the so-called "vital" phenomena are due to the fact that they constitute manifestations of exchange of energy between the phases of a heterogeneous system. Special degrees of activity may be detected during the transformation of energy, e. g., electric phenomena during the oxidation of phosphorus or benzaldehyde. Life is an incessant change, or a continuous transfer of energy, and a system in a state of statical equilibrium is equivalent to death. In "After the War," Ettore Ciccotti foresees that the causes of conflict between nations are too deeply rooted to be eliminated by this war, whatever may be its result. The hegemony of the money market will be tranferred from the Old World to the New. The experiences of the last two years

will force every nation to undertake an exhaustive examination of its natural resources, and all energies will be devoted to the development of productive forces, and to organization for the purpose of unifying, multiplying, and rendering immediately available the energies of the state. Social justice, emancipation from class domination, and the recognition of peace as the universal goal of humanity will be the chief articles in the creed of the new international socialist party. And finally we may see, for the next generation or so, a humanity penetrated by the most poignant of pessimisms.

The July number opens with a paper by Antonio Favaro on the "Effect of the Condemnation of Galileo upon the Progress of Science." One of the most serious consequences was the difficulty found by men like Descartes in the full expression of their thought. Rome was powerless to check the innermost thoughts of men, but she could and did use her powers of intimidation to such effect that what should have been the philosophy of the age was either directly checked, made but a timid advance, or was diverted from its natural channels. All that was new or out of the common rut came under suspicion. All the protestations and submissions of Descartes could not prevent his works being placed on the Index. At the nod of a Richelieu the Sorbonne returned to a sun revolving round the earth.—The study of the phenomena of cathodic bombardments, set forth in his article "The Colloids and Projections from Cathodes," has led Professor Houllevigue to the conclusion that the projectiles launched by an electrode of silver are of the same order of magnitude as the granules of colloidal silver deposited in the Bredig process. Experiments carried on for several years have brought him to the belief that it will clarify our ideas if we cease for the moment the study of colloids from the point of view of a solid or liquid state, and consider what takes place in the gaseous colloidal medium which surrounds the cathode in activity in a vacuum tube. This view he throws out with some reserve; but, as he reminds us, even if an hypothesis proves to be unfounded, it may still play its part in the progress of science by the experiments to which it leads.—Professor Lalande contributes a subtly conceived little paper on the "Relations between Logic and Psychology." The progress of logical intelligibility is marked by the discovery of resemblances in given differences. The ideal of scientific success is the absorption of facts *sui generis* in a wider formula common to

them all. We may not reach the why and the wherefore of the world by means of the logical norm, but the rich diversity of the universe provides for that norm, as it were, the fuel for the fire.—The "Reparation of the Waste of War," and the "Principal Economic Consequences of the Interruption of International Exchanges" form the texts for two articles by Mr. W. R. Scott and F. Virgilii respectively. Dr. Jankelevitch reviews the series of articles that have appeared in *Nature* and *Science Progress* dealing with the organization of science, its relations to the state, and the proper payment of scientific men.

In the August *Scientia* J. L. Heiberg discusses the role of Archimedes in the development of the exact sciences. The author describes the probable equipment with which Archimedes began his mathematical labors. His mastery of the weapons of his age in the attack on the theory of the conic sections, and their application to the solution of problems of a higher order, was considerable enough to win for Apollonius in later days the title of "plagiarist." The spiral of Archimedes was a magnificent geometrical effort which was later utilized in important investigations on the surface of the cylinder and sphere. The *Arenarius* reminds us of his success in dealing with large numbers. The influence of the great Greek upon succeeding ages is then carefully traced. The treatise on mechanical method, discovered but a decade ago,[1] would have greatly simplified the work of Kepler and Cavalieri had it been in their hands.—The "Hydrology of the Carso" of Istria, Carniola and Trieste, forms the subject of a most interesting geological paper by Prof. Luigi De Marchi.—A paper by Prof. L. Vialleton on the biogenetic law is based upon the precocity of the appearance of different types of the same group in the paleontological development. There is an undoubted parallelism between paleontological and ontogenetic development. Both issue at an early stage in well-defined and often divergent forms between which are no intermediaries. The anterior limb of the lemur could never be transformed into the wing of the bat, because its construction enables it to act in a vertical or nearly vertical plane, and never in the horizontal plane as in the case of the wing. There is little doubt that Cuvier's correlation law will play an important part in the explanation of the morphological puzzles that have yet to be unravelled.—Messrs. J. B.

[1] *Geometrical Solutions Derived from Mechanics.* Discovered and translated by Professor Heiberg. English edition published by Open Court Publishing Company, 1909.

Clark and E. Catellani treat respectively of the economic dynamics of war and the conditions under which peace may be secured and further outbreaks of war prevented.

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The number of *Science Progress* for April, 1916, contains papers by James Johnstone on the mathematical theory of organic variability, by David Fraser Harris on the specific characteristics of vitality, by C. Mansell Moullin on the natural history of tumors, and by Joseph Offord on the knowledge of the ancients regarding the propagation of disease by flies and rodents; and the third part of the investigations by Sir Ronald Ross on the solution of equations by operative division. Besides this there are very many reviews of books, notes, correspondence, and the usual long quarterly reports on the recent advances made in the various branches of science.

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With the July number of *Science Progress* a new volume begins—the eleventh—and an extension of purview is shown by the addition of "and Affairs" to the old title, "A Quarterly Review of Scientific Thought and Work." Articles no longer are awarded the bulk of the space at the disposal of the Editor. Just over three quarters of the number are given to notes, essays, reviews and to the very valuable pages entitled "Recent Advances in Science," now running to 50 pages or so. Mr. Bradford's "Historical Sketch of the Chemistry of Rubber" closes with an expression of confidence that before very long we shall have a synthetic rubber on the market. Mr. Friend deals with the "Bionomics of English Oligochaeta," Part ii—a most useful piece of (unpaid) work, in which stress is laid on the benignant role of Pachydrilids in the economy of nature. "A Biologist" in "The Pollution of the Sea" has an opportunity, of which he cordially avails himself, of exposing the mischiefs inherent in lawyer-made law upon matters dealing with the realities of life. And Mr. Reid Moir is at home in "Flint Fracture and Flint Implements," giving an account of experiments devised to distinguish between human and natural flaking. Among the essay-reviews is a long and interesting account of a great medical reformer—John Shaw Billings, "a man who was unique in the history of his profession." Φ

(or any other national language) as an international medium? And without such acceptance no language, natural or artificial, can become that medium.

A simplification of English spelling would not make English appreciably easier for foreigners; it would make it easier for English and American children who know the language already, but not for others. Besides, even the reformed spelling gives absolutely no clue how a word should be pronounced, unless one knows the word already. To take one or two examples from Mr. Man's own letter: Why should "been" and "in" be pronounced with a short *i* and spelt differently? Who can guess that in "sho" and "to," though written with the same vowel, that vowel is pronounced differently, etc., etc.

All this shows that only a "neutral" language, which also in its grammar, spelling and word-construction is easy, can ever hope to be accepted as "the international language."

C. T. STRAUSS.

LEIPSIC, GERMANY.

BOOK REVIEWS AND NOTES.

CONTRIBUTIONS TO THE FOUNDING OF THE THEORY OF TRANSFINITE NUMBERS. By *Georg Cantor*. Translated and provided with an Introduction by *Philip E. B. Jourdain*. Chicago and London: Open Court Publishing Company. Pages, 212, Price, $1.25.

Everybody knows and constantly uses the whole numbers, 1, 2, 3, and so on; and uses the word "infinite" for something which, like the above series of numbers, has no end. In fact, however large a number is, we can always think of a still larger one, and thus we never get to an end of the above series. But the great German mathematician Georg Cantor, who is still living at Halle, first saw about 1870 that in certain branches of mathematics we must contemplate a new series of numbers each of which is greater than any of the above finite numbers, and thus has a place *after all* the finite numbers; just as in the spectrum a shade of red has a place after all the innumerable shades of orange though we cannot say that there is a last shade of orange. Cantor spent years in getting himself and others accustomed to the strange idea of infinite or "transfinite" numbers, which, though each consisted of an unending set of units, could be thought of as complete wholes much as "all the points in the line AB" denotes an infinite set and can yet be treated as a completed whole. With this end in view Cantor studied deeply the arguments of philosophers, theologians and mathematicians about the infinite. At last, in 1895 and 1897, he succeeded in putting the results of nearly thirty years of work into a logical form which any intelligent person will not find very hard to understand. It is these famous essays that are here translated. In the introduction Mr. Jourdain has shown in detail how the new ideas grew from the work of Cantor's predecessors and in Cantor's own mind, and how these ideas must now be studied and used by *all philosophers, theologians, logicians, those interested in the foundations of the science of number and all mathematics*, and those who think about the ultimate constitution of space and matter, besides all mathematicians. This book appeals to any one who wants to understand one of the main things that has revolutionized many of the methods and problems and applications of modern mathematics and philosophy of mathematics and philosophy in general, and feels sympathy with those who want to know what numbers and fractions and space and matter are.

Why should mathematics interest everybody? Mere calculation is not interesting except to a few people. But even letting the mind rest on great and firm eternal truths is enchanting; living and working to find out more about them is absorbing. *Mathematics is one of the few paths to truth, and*

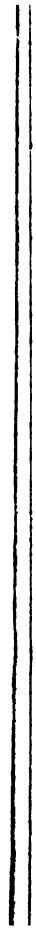
the search for truth is the religion of all thinking men and women nowadays. Mathematics is one of the most living of studies when treated historically so that we can follow the birth and development of great ideas. Thinking teachers know how attractive and indispensable it is to introduce students to new ideas and the truths they mirror, slowly and, if possible, as the actual discoverers were introduced to them. ϕ

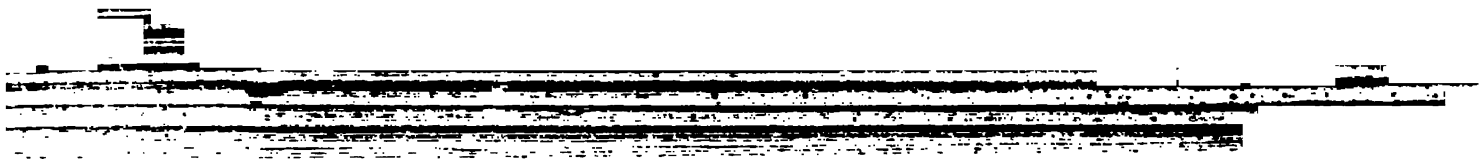
NAPIER TERCENTENARY MEMORIAL VOLUME. Edited by *Cargill Gilston Knott.* Published for the Royal Society of Edinburgh by Longmans, Green and Co., London and New York, 1915. Pp. xii, 441. Price $7 net or 21s. net.

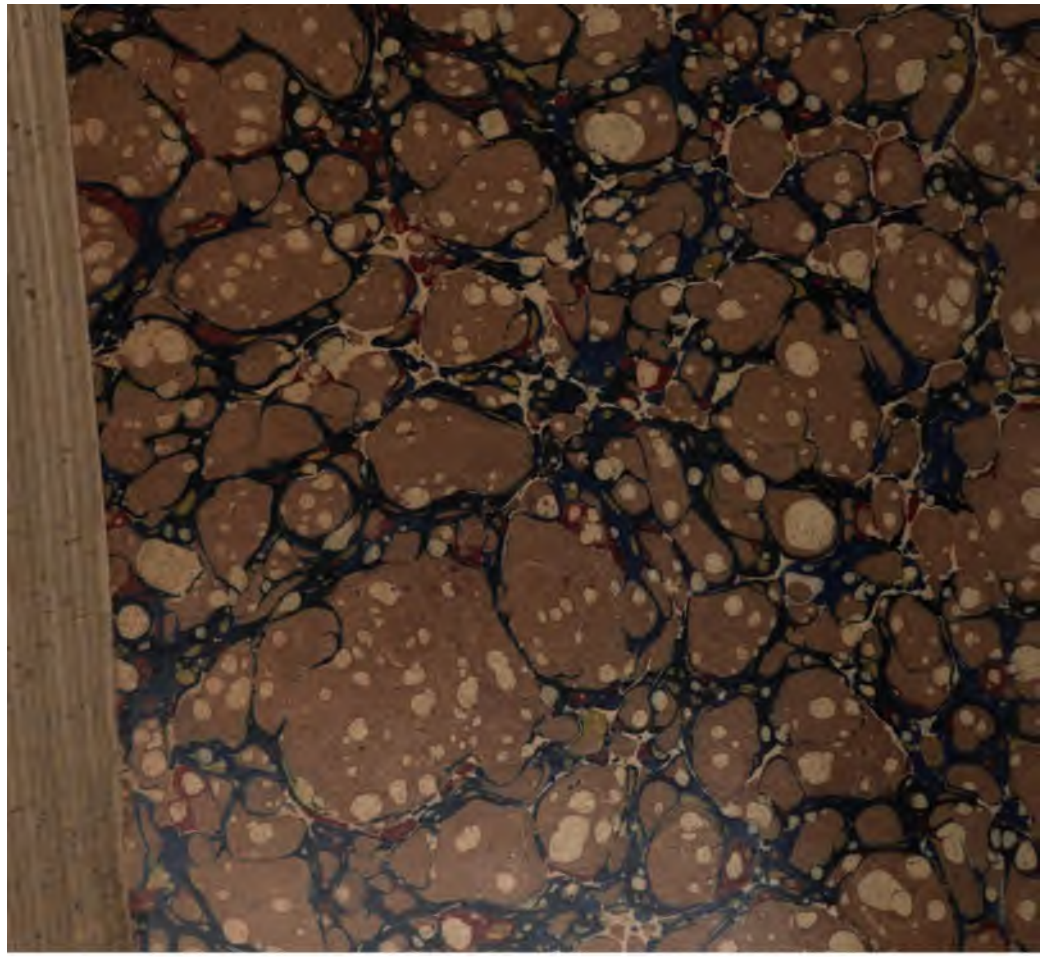
This magnificent volume contains the addresses and essays communicated to the international congress held at Edinburgh in July, 1914, in celebration of the tercentenary of the first publication of John Napier's system of logarithms. It is superbly printed and bound, contains a frontispiece in color from the well-known portrait of Napier in the University of Edinburgh and has several other plates. This congress, of which a full account is given by Dr. Knott, was the last international congress of any kind held before the European war broke out; and there is a certain melancholy interest in glancing through this volume and seeing contributions of great value not only from Great Britain but also from America, France, Germany, Italy, and even Turkey. The communications fall into two groups. Some treat of the life and work of Napier, and some with subsequent developments of the logarithmic idea and contain valuable additions to our means of calculation. But the greatest interest, perhaps, will center in the contributions of the first group, and of these the most striking is the inaugural address by Lord Moulton, in which an attempt is made to reconstruct the gradual evolution of Napier's great discovery. Most of us know that Lord Moulton, in his career at the Bar, had great experience in the study of inventions, and this address of his is one of the most important contributions to the history of mathematics that has been made in recent years. Indeed the whole volume is quite indispensable for the future historian of mathematics. We may mention that Prof. F. Cajori shows how the history of the subject has been mangled by authoritative historians of the past, and that there are also notable contributions made by Dr. J. W. L. Glaisher, Prof. D. E. Smith, Prof. G. A. Gibson, and many others. Finally it must be mentioned that a copy of the rare work of Bürgi was lent to the congress by the town library of Danzig and it is fully described in this volume. Φ

A COURSE OF MODERN ANALYSIS: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; with an Account of the Principal Transcendental Functions. By *E. T. Whittaker* and *G. N. Watson.* Second edition, completely revised. Pp. vi, 560. Cambridge (England): University Press, 1915. 18s. net.

The first edition (by Professor Whittaker alone) of this work was published in 1902, and in the preparation of the second edition Professor Whittaker has been most ably helped by Mr. Watson. To Mr. Watson the new chapters on Riemann Integration, Integral Equations, and the Riemann Zeta-Function







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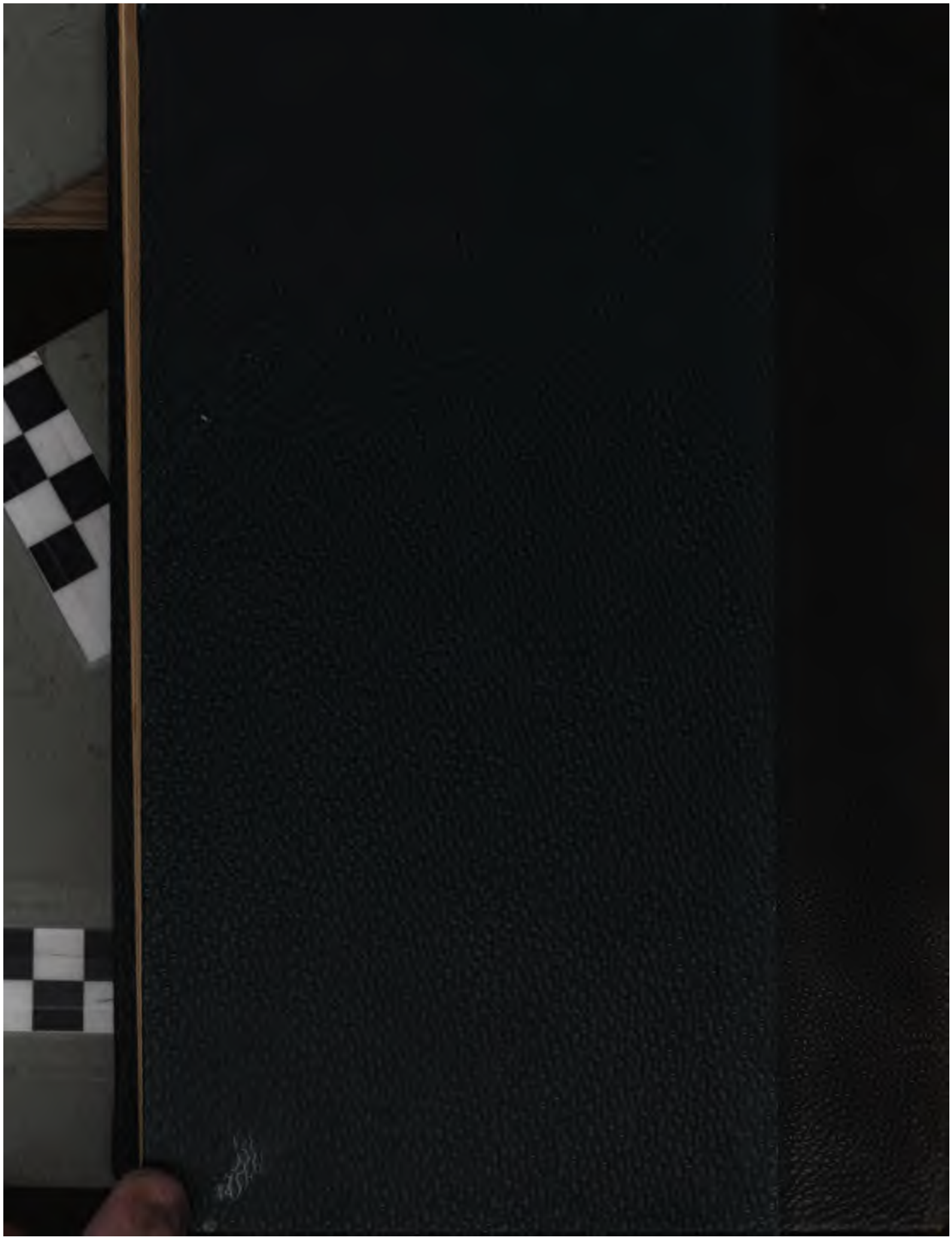
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